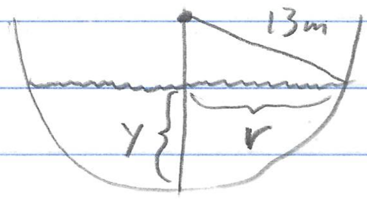
3.10.29 Water is flowing at the rate of $6 m^3/min$ from a reservoir shaped like a hemispherical bowl of radius 13 m, shown below. The volume of water in a hemispherical bowl of radius R is $V = (\pi/3)\, y^2 (3R - y)$ when the water is y meters deep.



(a) At what rate is the water level changing when the water is 8 m deep?

Solution

① We are given the picture above, where y is the depth of the water and r is the radius of the circular surface of the water.

② With V as the volume of the water, we are given $dV/dt = -6\ m^3/min$.

③ The question is $dy/dt = ?$ when $y = 8m$.

④ We are given the equation relating V and y (where $R = 13$ m):

$$V = \frac{\pi}{3} y^2 (3(13) - y) = \frac{\pi}{3} y^2 (39 - y).$$

⑤ Differentiating implicitly with respect to t:

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{\pi}{3}y^2(39-y)\right] = \frac{\pi}{3}\frac{d}{dt}\left[39y^2 - y^3\right]$$

or $$\frac{dV}{dt} = \frac{\pi}{3}\left(78y\left[\frac{dy}{dt}\right] - 3y^2\left[\frac{dy}{dt}\right]\right)$$

$$= \left(26\pi y + \pi y^2\right)\frac{dy}{dt}$$

or $$\frac{dy}{dt} = \frac{dV/dt}{26\pi y - \pi y^2}.$$

⑥ Evaluate: When $y = 8$ m and $dV/dt = -6\,m^3/min$ we have

$$\frac{dy}{dt} = \frac{-6\ m^3/min}{\left(26\pi(8) - \pi(8)^2\right)m^2} = \frac{-6}{208\pi - 64\pi}\ m/min$$

$$= \frac{-6}{144\pi}\ m/min = \boxed{\frac{-1}{24\pi}\ m/min}\ \square$$

(b) What is the radius r of the water's surface when the water is y meters deep?

Solution

From the picture above, we have the right triangle:

So by the Pythagorean Theorem we have

$$r^2 + (13-y)^2 = 13^2 \quad \text{or} \quad r^2 = 169 - (13-y)^2$$

or $$r^2 = 169 - 169 + 26y - y^2 = 26y - y^2.$$

So, since $r > 0$, $\boxed{r = \sqrt{26y - y^2}}$. $\square$

(c) At what rate is the radius r changing when the water is 8 m deep?

Solution

① We have the picture given above.

② We have that $dy/dt = \frac{-1}{24\pi}$ m/min when $y = 8$ m by part (a).

③ The question is $dr/dt = ?$ when $y = 8$m.

④ From part (b) we have the relationship between r and y of $r = (26y - y^2)^{1/2}$.

⑤ Differentiating implicitly with respect to t:

$$\frac{d}{dt}[r] = \frac{d}{dt}\left[(26y-y^2)^{1/2}\right]$$ or

$$\frac{dr}{dt} = \frac{1}{2}(26y - y^2)^{-1/2}\left[26\frac{dy}{dt} + 2y\left[\frac{dy}{dt}\right]\right]$$

$$= \frac{(13-y)}{\sqrt{26y-y^2}}\frac{dy}{dt}.$$

⑥ Evaluate: WHEN $y = 8$ m and $\frac{dy}{dt} = \frac{-1}{24\pi}$ m/min we have

$$\frac{dr}{dt} = \frac{13-(8)}{\sqrt{26(8)-(8)^2}}\left(\frac{-1}{24\pi}\right) \text{ m/min}$$

$$= \left(\frac{5}{\sqrt{144}}\right)\left(\frac{-1}{24\pi}\right) \text{ m/min} = \frac{-5}{(12)(24\pi)} \text{ m/min}$$

$$= \boxed{\frac{-5}{288\pi} \text{ m/min}}. \quad \square$$