3.10.31 A spherical balloon is inflated with helium at the rate of 100π ft^3/min.

(a) How fast is the balloon's radius increasing at the instant the radius is 5 ft?

(b) How fast is the surface area increasing then?

Solution

(a) ① Draw a picture: Let r be the radius of the sphere and V its volume.

← 100π ft^3/min

② We know $\dfrac{dV}{dt} = 100\pi$ ft^3/min.

③ The question is $\dfrac{dr}{dt} = ?$ when $r = 5$ ft.

④ A relationship between V and r (for a sphere) is $V = \frac{4}{3}\pi r^3$.

⑤ Differentiating implicitly with respect to time t, we have

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] \text{ or } \frac{dV}{dt} = \frac{4}{3}\pi\left[3r^2\left[\frac{dr}{dt}\right]\right]$$

$$\text{or } \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \text{ or } \frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}.$$

⑥ Evaluate: WHEN $r = 5$ ft and $dV/dt = 100\pi$ ft^3/min, then

$$\frac{dr}{dt} = \frac{(100\pi \text{ ft}^3/\text{min})}{4\pi(5\text{ ft})^2} = \boxed{1 \text{ ft/min}}.$$

(b) ① We have the picture in part (a).

② We know $\frac{dV}{dt} = 100\pi \text{ ft}^3/\text{min}$ and $\frac{dr}{dt} = 1 \text{ ft}/\text{min}$ by part (a).

③ The question is $\frac{dS}{dt} = ?$ where S is the surface area.

④ A relationship between S and r (for a sphere) is $S = 4\pi r^2$.

⑤ Differentiating implicitly with respect to time t, we have

$$\frac{d}{dt}[S] = \frac{d}{dt}[4\pi r^2] \text{ or } \frac{dS}{dt} = 4\pi\left[2r\left[\frac{dr}{dt}\right]\right]$$

or $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$.

⑥ Evaluate: WHEN $r = 5$ ft and $\frac{dr}{dt} = 1$ ft/min,

$$\frac{dS}{dt} = 8\pi(5 \text{ ft})(1 \text{ ft}/\text{min}) = \boxed{40\pi \text{ ft}^2/\text{min}}. \square$$