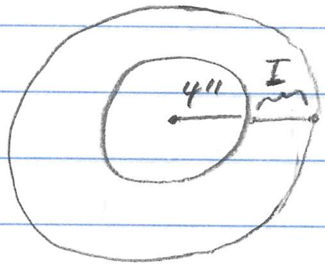
3.10.41 A spherical iron ball 8 inches in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 10 in^3/min, how fast is the thickness of the ice decreasing when it is 2 inches thick? How fast is the outer surface area of ice decreasing at this point in time?

Solution

① A picture is:



Let I be the thickness of the ice. Let V be the volume of the sphere determined by the iron ball and the layer of ice (so the radius is 4 + I inches, and $V = \frac{4}{3}\pi(4+I)^3$.

② We have

$$\frac{dV}{dt} = -10 \text{ in}^3/\text{min}$$

③ The question is $\frac{dI}{dt} = ?$ when $I = 2$ in.

④ The variables (I and V) are related as

$$V = \frac{4}{3}\pi(4+I)^3.$$

(5) Differentiate implicitly with respect to time:

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{4}{3}\pi(4+I)^3\right]$$

$$\frac{dV}{dt} = \frac{4}{3}\pi\left[3(4+I)^2\left[0+\frac{dI}{dt}\right]\right]$$

$$= 4\pi(4+I)^2\frac{dI}{dt}.$$

(6) Evaluate: WHEN $I = 2$ in and $\frac{dV}{dt} = 10\text{ in}^3/\text{min}$ we have

$$\frac{dV}{dt} = (-10\text{ in}^3/\text{min}) = 4\pi(4+(2\text{ in}))^2\frac{dI}{dt}.$$

$$\frac{dI}{dt} = \frac{-10\text{ in}^3/\text{min}}{144\pi\text{ in}^2}$$

or $$\boxed{\frac{dI}{dt} = \frac{-5}{72\pi}\text{ in/min.}}$$ □

ALSO: How fast is the outer surface area decreasing then?

Solution

(2) We have: $S = 4\pi r^2$ is the surface area of a sphere of radius r.

Here, the radius is $4 + I$ in, so

$$\begin{pmatrix}\text{surface}\\ \text{area}\end{pmatrix} = S = 4\pi(4+I)^2.$$

③ The question is $\frac{dS}{dt} = ?$ when $I = 2$ in.

④ The variable S and I are related as

$$S = 4\pi(4+I)^2.$$

⑤ Differentiate:

$$\frac{d}{dt}[S] = \frac{d}{dt}\left[4\pi(4+I)^2\right]$$

$$\frac{dS}{dt} = 4\pi\left[2(4+I)^2\left[\frac{dI}{dt}\right]\right]$$

or $$\frac{dS}{dt} = 8\pi(4+I)\frac{dI}{dt}.$$

⑥ Evaluate: WHEN $I = 2$ inches and (by the first part of the question) $dI/dt = -5/(72\pi)$ in/min, we have

$$\frac{dS}{dt} = 8\pi(4+(2))\left(\frac{-5}{72\pi}\right) \text{ in}^2/\text{min}$$

$$= \frac{-30}{9} \text{ in}^2/\text{min} = \boxed{\frac{-10}{3} \text{ in}^2/\text{min}}$$ □