3.2.12 Find $\frac{dz}{dw}$ if $z(w) = \frac{1}{\sqrt{w^2-1}}$.

Solution

By definition,

$$\frac{dz}{dw} = \lim_{h\to 0} \frac{z(w+h) - z(w)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{(w+h)^2+1}} - \frac{1}{\sqrt{w^2-1}}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sqrt{w^2-1} - \sqrt{(w+h)^2+1}}{\sqrt{(w+h)^2+1}\sqrt{w^2-1}}\right)$$ getting a common denominator

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sqrt{w^2-1} - \sqrt{(w+h)^2+1}}{\sqrt{(w+h)^2+1}\sqrt{w^2-1}}\right)\left(\frac{\sqrt{w^2-1} + \sqrt{(w+h)^2+1}}{\sqrt{(w+h)^2+1} + \sqrt{w^2-1}}\right)$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\left(\sqrt{w^2-1}\right)^2 - \left(\sqrt{(w+h)^2+1}\right)^2}{\sqrt{(w+h)^2+1}\sqrt{w^2-1}\left(\sqrt{(w+h)^2+1} + \sqrt{w^2-1}\right)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{(w^2-1) - (w^2+2wh+h^2+1)}{\sqrt{(w+h)^2+1}\sqrt{w^2-1}\left(\sqrt{(w+h)^2+1} + \sqrt{w^2-1}\right)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{-2wh-h^2}{\sqrt{(w+h)^2+1}\sqrt{w^2-1}\left(\sqrt{(w+h)^2+1} + \sqrt{w^2-1}\right)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{h(-2w-h^2)}{\sqrt{(w+h)^2+1}\sqrt{w^2-1}\left(\sqrt{(w+h)^2+1} + \sqrt{w^2-1}\right)}$$

$$= \lim_{h\to 0} \frac{-2w-h^2}{\sqrt{(w+h)^2+1}\sqrt{w^2-1}\left(\sqrt{(w+h)^2+1} + \sqrt{w^2-1}\right)}$$

$$= \frac{-2w-(0)^2}{\sqrt{(w+(0))^2+1}\,\sqrt{w^2-1}\left(\sqrt{(w+(0))^2+1}+\sqrt{w^2-1}\right)}$$

$$= \frac{-2w}{(w^2-1)\left(2\sqrt{w^2-1}\right)} = \boxed{\frac{-w}{(w^2-1)^{3/2}}}. \quad \square$$