3.2.17 Differentiate $y = f(x) = \dfrac{8}{\sqrt{x-2}}$. Then find an equation of tangent line at $(x, y) = (6, 4)$.

Solution

We need

$$y' = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\dfrac{8}{\sqrt{(x+h)-2}} - \dfrac{8}{\sqrt{x-2}}\right)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{8\sqrt{x-2} - 8\sqrt{x+h-2}}{\sqrt{x+h-2}\,\sqrt{x-2}}\right)$$

$$= \lim_{h\to 0} \frac{8}{h}\left(\frac{\sqrt{x-2} - \sqrt{x+h-2}}{\sqrt{x+h-2}\,\sqrt{x-2}}\right)\left(\frac{\sqrt{x-2} + \sqrt{x+h-2}}{\sqrt{x-2} + \sqrt{x+h-2}}\right)$$

$$= \lim_{h\to 0} \frac{8}{h} \frac{(\sqrt{x-2})^2 - (\sqrt{x+h-2})^2}{\sqrt{x+h-2}\,\sqrt{x-2}\,(\sqrt{x-2} + \sqrt{x+h-2})}$$

$$= \lim_{h\to 0} \frac{8}{h} \frac{(x-2) - (x+h-2)}{\sqrt{x+h-2}\,\sqrt{x-2}\,(\sqrt{x-2} + \sqrt{x+h-2})}$$

$$\overset{F}{=} \lim_{h\to 0} \frac{8}{h} \frac{-h}{\sqrt{x+h-2}\,\sqrt{x-2}\,(\sqrt{x-2} + \sqrt{x+h-2})}$$

$$\overset{C}{=} \lim_{h\to 0} \frac{-8}{\sqrt{x+h-2}\,\sqrt{x-2}\,(\sqrt{x-2} + \sqrt{x+h-2})}$$

$$\stackrel{S}{=} \frac{-8}{\sqrt{x+(0)-2}\,\sqrt{x-2}\,\left(\sqrt{x-2}+\sqrt{x+(0)-2}\right)}$$

$$= \frac{-8}{\left(\sqrt{x-2}\right)^2\left(2\sqrt{x-2}\right)} = \boxed{\frac{-4}{(x-2)^{3/2}}}.$$

Now, the derivative gives slopes of tangent lines at various points. At point $(x,y) = (6,4)$, the slope of a tangent line is

$$m = f'(6) = \frac{-4}{((6)-2)^{3/2}} = \frac{-4}{4^{3/2}}$$

$$= \frac{-4}{4\sqrt{4}} = \frac{-1}{2}.$$

So the desired line is (by the point-slope formula):

$$y-(4) = \left(\frac{-1}{2}\right)(x-6)$$

or $y - 4 = \frac{-1}{2}x + 3$

or $\boxed{y = \frac{-1}{2}x + 7}$. □