3.2.41 Determine if the piecewise-defined function is differentiable at the origin:

$$f(x) = \begin{cases} 2x-1, & x \geq 0 \\ x^2+2x-7, & x < 0. \end{cases}$$

<u>Solution</u>

Since f is defined piecewise, we consider one-sided limits in the definition of derivative (in essence, we consider left-hand and right-hand derivatives at $x=0$). We have the one-sided limits when addressing $f'(0)$:

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{((h)^2+2(h)-7)-(2(0)-1)}{h}$$

since $0+h = h < 0$ here

$$= \lim_{h\to 0^-} \frac{h^2+2h-6}{h} = \lim_{h\to 0^-}\left(h+2-\frac{6}{h}\right) = \infty$$

since $\displaystyle\lim_{h\to 0^-} \frac{-6}{h} = -6 \lim_{h\to 0^-}\left(\frac{1}{h}\right) = -(-\infty) = \infty$;

see Dr. Bob's infinite limits theorem.

Also,

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{(2(h)-1)-(2(0)-1)}{h}$$

since $0+h = h \geq 0$ here

$$= \lim_{h\to 0^+} \frac{2h}{h} = \lim_{h\to 0^+} 2 = 2.$$

Since the one-sided limits are different then the (two-sided) limit,

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h},$$

does not exist by Theorem 2.6 ("Relation Between One-Sided and Two-Sided Limits).

That is, the function is $\boxed{\text{NOT differentiable}}$ at the origin (where $x = 0$). $\square$