3.3.57 Find the equations of the tangent lines to $y = f(x) = \dfrac{4x}{x^2+1}$ at the origin (i.e., the point $(0,0)$) and the point $(1,2)$.

Solution

Recall that the derivative $y' = f'(x)$ gives the slope of a line tangent to $y = f(x)$ as a function of x. Next,

$$y' = f'(x) = \frac{[4](x^2+1) - (4x)[2x]}{(x^2+1)^2}$$

$$= \frac{4x^2+4-8x^2}{(x^2+1)^2} = \frac{-4x^2+4}{(x^2+1)^2}.$$

So at the origin $(0,0)$ the slope of a tangent line is $m_0 = f'(0) = \dfrac{-4(0)^2+4}{((0)^2+1)^2} = 4$.

By the point slope formula, the tangent line at the origin is "$y - y_1 = m(x - x_1)$" or $y - 0 = 4(x-0)$ or $\boxed{y = 4x}$.

At the point $(1,2)$, the slope is $f'(1) = \dfrac{-4(1)+4}{((1)^2+1)^2} = 0 = m_1$. Therefore, "$y - y_1 = m(x - x_1)$" or $y - 2 = 0(x-1)$ or $\boxed{y = 2}$. □