3.6.105 Temperature and the Period of a Pendulum

For oscillations of small amplitude (short swings), we may safely model the relationship between the period T and the length L of a simple pendulum with the equation $T = 2\pi\sqrt{L/g}$, where g is the constant acceleration of gravity at the pendulum's location. If we measure g in centimeters per second squared, we measure L in centimeters and T in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L. In symbols, with u being temperature and k the proportionality constant, $\frac{dL}{du} = kL$. Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

Solution

The rate at which the period T changes with respect to temperature u is dT/du. By the Chain Rule (Theorem 3.2) we have

$$\frac{dT}{du} = \frac{dT}{dL}\frac{dL}{du}.$$

Since $T = 2\pi\sqrt{L/g} = \frac{2\pi}{\sqrt{g}}\sqrt{L} = \frac{2\pi}{\sqrt{g}} L^{1/2}$

then $\frac{dT}{dL} = \frac{2\pi}{\sqrt{g}}\left[\frac{1}{2} L^{-1/2}\right] = \frac{\pi}{\sqrt{gL}}$.

So $\frac{dT}{du} = \frac{dT}{dL}\frac{dL}{du} = \left(\frac{\pi}{\sqrt{gL}}\right)(kL)$

$= k\frac{\pi\sqrt{L}}{\sqrt{g}} = \frac{k}{2}\left(2\pi\sqrt{\frac{L}{g}}\right)$

$= \frac{kT}{2}$ since $T = 2\pi\sqrt{\frac{L}{g}}$,

as claimed. $\square$