3.6.89 Suppose that functions f and g and their derivatives with respect to x have the following values at $x=2$ and $x=3$:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	-3
3	3	-4	2π	5

evaluated at

(e) $\dfrac{d}{dx}\left[f(g(x))\right]\Big|_{x=2} = ?$

Solution

By the Chain Rule (Theorem 3.2) we have

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\left[g'(x)\right].$$

So we want $f'(g(2))\,g'(2)$.

We have $g(2) = 2$, $g'(2) = -3$, and $f'(2) = 1/3$, so

$$f'(g(2))\,g'(2) = f'(2)(-3) = \left(\tfrac{1}{3}\right)(-3) = \boxed{-1}.$$

(f) $\left.\frac{d}{dx}\left[\sqrt{f(x)}\right]\right|_{x=2} = ?$

Solution

We have

$$\frac{d}{dx}\left[\sqrt{f(x)}\right] = \frac{d}{dx}\left[(f(x))^{1/2}\right]$$

$$= \frac{1}{2}(f(x))^{-1/2}\left[f'(x)\right].$$

so,

$$\left.\frac{d}{dx}\left[\sqrt{f(x)}\right]\right|_{x=2} = \frac{1}{2}(f(2))^{-1/2} f'(2)$$

$$= \frac{1}{2}(8)^{-1/2}\left(\frac{1}{3}\right) \quad \text{since } f(2) = 8 \text{ and } f'(2) = 1/3$$

$$= \frac{1}{6\sqrt{8}} = \frac{1}{6\sqrt{4\cdot 2}} = \boxed{\frac{1}{12\sqrt{2}}}. \quad \square$$

(g) $\left.\frac{d}{dx}\left[\frac{1}{g^2(x)}\right]\right|_{x=3} = ?$

Solution

We have

$$\frac{d}{dx}\left[\frac{1}{g^2(x)}\right] = \frac{d}{dx}\left[\left(\frac{1}{g(x)}\right)^2\right] = \frac{d}{dx}\left[(g(x))^{-2}\right]$$

$$= -2(g(x))^{-3}\,[g'(x)].$$

so

$$\frac{d}{dx}\left[\frac{1}{g^2(x)}\right]\Bigg|_{x=3} = -2(g(3))^{-3}[g'(3)]$$

$$= -2(-4)^{-3}(5) = \frac{-10}{-64} = \boxed{\frac{5}{32}}$$

since $g(3) = -4$ and $g'(3) = 5$. □

(h) $\displaystyle \frac{d}{dx}\left[\sqrt{f^2(x) + g^2(x)}\right]\Bigg|_{x=2} = ?$

Solution

We have $\displaystyle \frac{d}{dx}\left[\sqrt{f^2(x)+g^2(x)}\right] = \frac{d}{dx}\left[\left((f(x))^2+(g(x))^2\right)^{1/2}\right]$

$$= \frac{1}{2}\left((f(x))^2+(g(x))^2\right)^{-1/2}\left[2(f(x))[f'(x)]+2(g(x))[g'(x)]\right]$$

$$= \frac{f(x)\,f'(x) + g(x)\,g'(x)}{\sqrt{(f(x))^2+(g(x))^2}}.$$

So $\displaystyle \frac{d}{dx}\left[\sqrt{f^2(x)+g^2(x)}\right]\Bigg|_{x=2} = \frac{(8)(1/3)+(2)(-3)}{\sqrt{(8)^2+(2)^2}}$

$$= \frac{8/3-6}{\sqrt{68}} = \frac{-10/3}{2\sqrt{17}} = \boxed{\frac{-5}{3\sqrt{17}}}$$ □