3.7.41 Consider $y = 2\sin(\pi x - y)$. Find the lines that are (a) tangent to, and (b) normal to the curve at the point $(1, 0)$.

Solution

(a) Well, the slope of a tangent line to a curve is given by $dy/dx = y'$. We have

$$\frac{d}{dx}[y] = \frac{d}{dx}[2\sin(\pi x - y)]$$

or $$y' = 2\cos(\pi x - y)[\pi - y']$$

or $$y' = 2\pi\cos(\pi x - y) - 2y'\cos(\pi x - y)$$

or $$y' + 2y'\cos(\pi x - y) = 2\pi\cos(\pi x - y)$$

or $$y'(1 + 2\cos(\pi x - y)) = 2\pi\cos(\pi x - y)$$

or $$y' = \frac{2\pi\cos(\pi x - y)}{1 + 2\cos(\pi x - y)}.$$

So, at the point $(x, y) = (1, 0)$ we have a tangent line with slope

$$m = y'\Big|_{(1,0)} = \frac{2\pi\cos(\pi(1) - (0))}{1 + 2\cos(\pi(1) - (0))}$$

$$= \frac{2\pi\cos(\pi)}{1 + 2\cos(\pi)} = \frac{2\pi(-1)}{1 + 2(-1)} = 2\pi$$

So the tangent line has slope $m = 2\pi$ and contains the point $(1,0)$. By the point-slope formula for a line, the tangent line is "$y - y_1 = m(x - x_1)$":

$$y - (0) = 2\pi(x - (1)) \text{ or } \boxed{y = 2\pi x - 2\pi}.$$

(b) The normal line to a curve is perpendicular to the tangent line. So the slope of the normal line at the point $(1, 0)$ is $m_\perp = -1/m = -1/(2\pi)$. So the normal line is

$$y - (0) = \frac{-1}{2\pi}(x - (1))$$

$$\text{or } \boxed{y = \frac{-1}{2\pi}x + \frac{1}{2\pi}}. \quad \square$$