3.9.45 Find $\frac{dy}{dx}$ at the point $P(\frac{1}{2}, -\sqrt{2})$

for $y \cos^{-1}(xy) = \frac{-3\sqrt{2}}{4}\pi$.

Solution

We differentiate implicitly with respect to x:

$$\frac{d}{dx}\left[y \cdot \cos^{-1}(xy)\right] = \frac{d}{dx}\left[\frac{-3\sqrt{2}}{4}\pi\right]$$

$$\left[\frac{dy}{dx}\right]\left(\cos^{-1}(xy)\right) + (y)\left[\frac{-1}{\sqrt{1-(xy)^2}}\left[[1](y)+(x)\left[\frac{dy}{dx}\right]\right]\right] = 0$$

or $$\frac{dy}{dx}\cos^{-1}(xy) + \frac{-y^2}{\sqrt{1-(xy)^2}} + \frac{-xy}{\sqrt{1-(xy)^2}}\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx}\left(\cos^{-1}(xy) - \frac{xy}{\sqrt{1-(xy)^2}}\right) = \frac{y^2}{\sqrt{1-(xy)^2}}.$$

or $$\frac{dy}{dx} = \frac{y^2/\sqrt{1-(xy)^2}}{\cos^{-1}(xy) - xy/\sqrt{1-(xy)^2}}$$

or $$\boxed{\frac{dy}{dx} = \frac{y^2}{\sqrt{1-(xy)^2}\cos^{-1}(xy) - xy}.}$$

At the point $P(\frac{1}{2}, -\sqrt{2})$ (i.e., at $x = 1/2$ and $y = -\sqrt{2}$) we have;

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1/2,-\sqrt{2})}$$

$$= \frac{(-\sqrt{2})^2}{\sqrt{1-((\frac{1}{2})(-\sqrt{2}))^2}\cos^{-1}((\frac{1}{2})(-\sqrt{2})) - (\frac{1}{2})(-\sqrt{2})}$$

$$= \frac{2}{\sqrt{1-(-\frac{\sqrt{2}}{2})^2}\cos^{-1}(\frac{-\sqrt{2}}{2}) + \frac{\sqrt{2}}{2}}$$

$$= \frac{2}{\sqrt{\frac{1}{2}}\cos^{-1}(\frac{-\sqrt{2}}{2}) + \frac{\sqrt{2}}{2}}$$

$$= \frac{2}{(\frac{\sqrt{2}}{2})(\frac{3\pi}{4}) + \frac{\sqrt{2}}{2}}$$ since $\sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$

and $\cos^{-1}(\frac{-\sqrt{2}}{2}) = \frac{3\pi}{4}$

$$= \frac{16}{3\pi\sqrt{2} + 4\sqrt{2}} = \frac{16\sqrt{2}}{6\pi + 8}$$

$$= \boxed{\frac{8\sqrt{2}}{3\pi+4}}. \square$$