4.1.59 Consider $y = f(x) = x\sqrt{4-x^2}$. Find the domain, the critical points, the local extrema, and absolute extrema.

Solution

Well, the domain of $f(x) = x\sqrt{4-x^2}$ is all x such that $4 - x^2 \geq 0$ (to avoid square roots of negatives). We have then $4 - x^2 \geq 0$ or $4 \geq x^2$ or $x^2 \leq 4$. Since the square root function is increasing then it preserves inequalities and hence $\sqrt{x^2} \leq \sqrt{4}$ or $|x| \leq 2$ or $-2 \leq x \leq 2$. Hence the $\boxed{\text{domain of } f \text{ is } [-2, 2].}$

Next for critical points consider $f(x) = x\sqrt{4-x^2} = x(4-x^2)^{1/2}$ and

$$f'(x) = [1]\left((4-x^2)^{1/2}\right) + (x)\left[\tfrac{1}{2}(4-x^2)^{-1/2}[-2x]\right]$$

$$= \sqrt{4-x^2} - x^2\frac{1}{\sqrt{4-x^2}}$$

$$= \frac{(\sqrt{4-x^2})^2 - x^2}{\sqrt{4-x^2}} = \frac{(4-x^2) - x^2}{\sqrt{4-x^2}}$$

$$= \frac{4-2x^2}{\sqrt{4-x^2}} = \frac{2(2-x^2)}{\sqrt{4-x^2}}.$$

With $f'(x) = \dfrac{2(2-x^2)}{\sqrt{4-x^2}}$ we have

$$f'(x) = \frac{2(2-x^2)}{\sqrt{4-x^2}} = 0 \text{ when } 2-x^2=0$$

or $x^2 = 2$ or $x = \pm\sqrt{2}$. So

$\boxed{x = \pm\sqrt{2} \text{ are critical points.}}$

Also, for $x \in [2, 2]$ we see that

$$f'(x) = \frac{2(2-x^2)}{\sqrt{4-x^2}} = \frac{2(2-x^2)}{\sqrt{(2-x)(2+x)}}$$

is undefined at $x = -2$ and $x = 2$ (since these x-values give division by 0 in $f'(x)$. That is,

$\boxed{x = \pm 2 \text{ are critical points.}}$

For the extrema (by Theorem 4.2 and related comments) we check the critical points and endpoints:

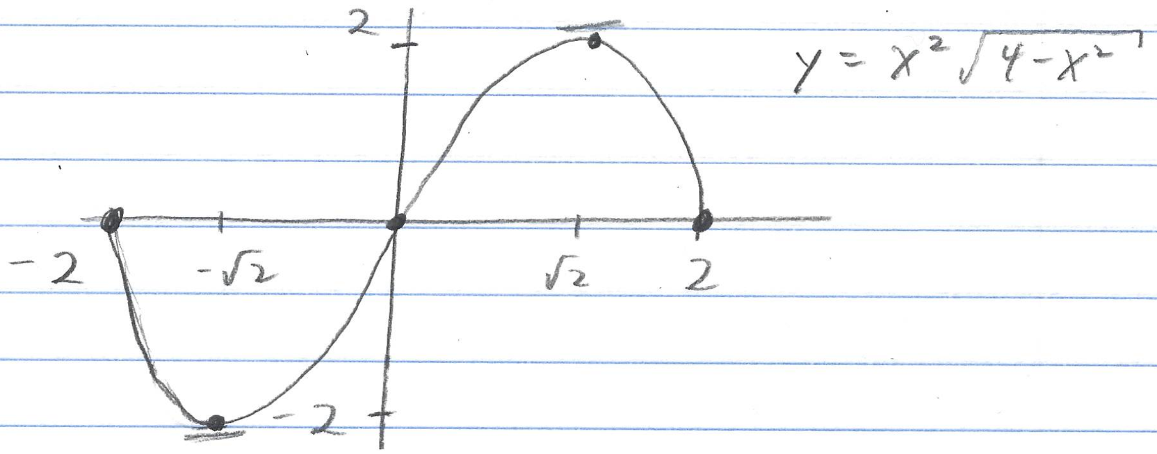
x	-2	$-\sqrt{2}$
$f(x)$	$(-2)\sqrt{4-(-2)^2} = 0$	$(-\sqrt{2})\sqrt{4-(-\sqrt{2})^2} = -\sqrt{2}(\sqrt{2}) = -2$

Continued →

x	$\sqrt{2}$	2
$f(x)$	$(\sqrt{2})\sqrt{4-(\sqrt{2})^2}$ $\sqrt{2}\sqrt{2} = 2$	$(2)\sqrt{4-(2)^2}$ $= 0$

$\boxed{\text{So } f \text{ has an absolute MAX of } 2 \text{ at } x=\sqrt{2} \text{ and } f \text{ has an absolute MIN of } -2 \text{ at } x=-\sqrt{2}.}$

The graph is (sort of...)



$\boxed{\text{Also, } f \text{ has a local MAX of } 0 \text{ at } x=-2 \text{ and } f \text{ has a local MIN of } 0 \text{ at } x=2.}$

$\square$