4.2.5 Find value $c \in [-1, 1]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{for } [a,b] = [-1,1],$$

as guaranteed to exist by Mean Value Theorem, for $f(x) = \sin^{-1}(x)$.

Solution:

Notice $f(x) = \sin^{-1}(x)$ is continuous on $[-1, 1]$ (consider the graph) and

$$f'(x) = \frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}} \quad \text{(see Section 3.9)}$$

so that f is differentiable on $(-1, 1)$. Hence, the hypotheses of the Mean Value Theorem are satisfied! Consider

$$f'(c) = \frac{1}{\sqrt{1-(c)^2}} = \frac{\sin^{-1}(1) - \sin^{-1}(-1)}{(1)-(-1)}$$

or $$\frac{1}{\sqrt{1-c^2}} = \frac{(\pi/2) - (-\pi/2)}{2} = \frac{\pi}{2}$$

or $$\sqrt{1-c^2} = \frac{2}{\pi} \quad \text{or} \quad \left(\sqrt{1-c^2}\right)^2 = \left(\frac{2}{\pi}\right)^2$$

or $$1 - c^2 = \frac{4}{\pi^2} \quad \text{or} \quad c^2 = 1 - \frac{4}{\pi^2} = \frac{\pi^2 - 4}{\pi^2}$$

or $$\sqrt{c^2} = \sqrt{\frac{\pi^2-4}{\pi^2}} \quad \text{or} \quad |c| = \frac{\sqrt{\pi^2-4}}{\pi},$$

so $$\boxed{c = \frac{\sqrt{\pi^2-4}}{\pi} \quad \text{or} \quad c = \frac{-\sqrt{\pi^2-4}}{\pi}} \quad \square$$