4.2.65 Show that $|\cos x - 1| \le |x|$ for all $x \in \mathbb{R}$.
HINT: Consider $f(t) = \cos t$ on $[0, x]$.

Solution

Consider $f(t) = \cos(t)$ on $[0, x]$ where $x > 0$. Then f is continuous on $[0, x]$ and $f'(t) = -\sin(t)$ exists on $(0, x)$. That is, f satisfies the hypotheses of the Mean Value Theorem! So, by the Mean Value Theorem there is $c \in (0, x)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$ where $f(t) = \cos(t)$, $f'(t) = -\sin(t)$, $a = 0$, and $b = x$.

So

$$-\sin(c) = \frac{\cos(x) - \cos(0)}{x - 0} = \frac{\cos(x) - 1}{x}.$$

Hence $\cos(x) - 1 = -x \sin(c)$ and

$$\begin{aligned} |\cos(x) - 1| &= |-x \sin(c)| = |x|\,|\sin(c)| \\ &\le |x|\,(1) \quad \text{since } |\sin(c)| \le 1 \\ &= |x|. \end{aligned}$$

Therefore $|\cos(x) - 1| \le |x|$ for $x > 0$.

Next, for $x < 0$ we consider $f(t) = \cos(t)$ on $[x, 0]$...

By the Mean Value Theorem again, there is $c \in (x, 0)$ such that

$$-\sin(c) = \frac{\cos(0) - \cos(x)}{0 - x} = \frac{1 - \cos(x)}{-x} = \frac{\cos(x) - 1}{x}.$$

Hence $\cos(x) - 1 = -x \sin(c)$ and

$$\begin{aligned} |\cos(x) - 1| &= |-x \sin(c)| = |x|\,|\sin(c)| \\ &\leq |x|(1) \quad \text{since } |\sin(c)| \leq 1 \\ &= |x|. \end{aligned}$$

Therefore $|\cos(x) - 1| \leq |x|$ for $x \in \mathbb{R}$.

Finally, if $x = 0$ then

$$0 = |\cos(0) - 1| \leq |0| = 0,$$ so that

$|\cos(x) - 1| \leq |x|$ for $x = 0$.

Therefore, $|\cos(x) - 1| \leq |x|$ for all $x \in \mathbb{R}$, as claimed. $\square$