4.2.73 Let f be differentiable at every value of x and suppose that $f(1) = 1$, that $f' < 0$ on $(-\infty, 1)$ and $f' > 0$ on $(1, \infty)$.

(a) Show that $f(x) \geq 1$ for all $x \in \mathbb{R}$.

Solution

ASSUME $f(x_0) < 1$ for some $x_0 \in (-\infty, 1)$. Then consider f on $[x_0, 1]$. Since f is differentiable at every value of x then by Theorem 3.1 (Differentiability Implies Continuity), f is continuous at every value of x. So the hypotheses of the Mean Value Theorem are satisfied and hence there is $c \in (x_0, 1)$ such that

$$f'(c) = \frac{f(1) - f(x_0)}{1 - x_0} = \frac{1 - f(x_0)}{1 - x_0}.$$

Since $x_0 < 1$ then $1 - x_0 > 0$ and since we assumed $f(x_0) < 1$ then $1 - f(x_0) > 0$, and hence

$$\frac{1 - f(x_0)}{1 - x_0} = f'(c) > 0.$$

But this is a CONTRADICTION since $c \in (x_0, 1)$ and $f' < 0$ on $(-\infty, 1)$ (in particular, $f'(c) < 0$). So the assumption that $f(x_0) < 1$ must be false and hence $f(x_0) \geq 1$ for all $x_0 \in (-\infty, 1)$. That is, $f(x) \geq 1$ for $x < 1$.

Similarly, ASSUME $f(x_0) < 1$ for some $x_0 \in (1, \infty)$. Then consider f on $[1, x_0]$. As above, by the Mean Value Theorem we have that there is $c \in (1, x_0)$ such that

$$f'(c) = \frac{f(x_0) - f(1)}{x_0 - 1} = \frac{f(x_0) - 1}{x_0 - 1}.$$

Since $x_0 > 1$ then $x_0 - 1 > 0$ and since we assumed $f(x_0) < 1$ then $f(x_0) - 1 < 0$, and hence

$$\frac{f(x_0) - 1}{x_0 - 1} = f'(c) < 0.$$

But this is a CONTRADICTION since $c \in (1, x_0)$ and $f' > 0$ on $(1, \infty)$ (in particular, $f'(c) > 0$). So the assumption that $f(x_0) < 1$ must be false and hence $f(x_0) \geq 1$ for all $x_0 \in (-\infty, 1)$. That is, $f(x) \geq 1$ for $x > 1$.

Finally, $f(1) = 1$ so $f(x) \geq 1$ for $x = 1$. That is, $f(x) \geq 1$ for all $x \in \mathbb{R}$. □

(b) Must $f'(1) = 0$? Explain.

Solution

By part (a), $f(x) \geq 1$ for all x, and $f(1) = 1$ by hypothesis. So f has a local minimum at $x = 1$ (in fact, it is an absolute minimum). Since 1 is an interior point of the domain, then by Theorem 4.2 (First Derivative Theorem for Local Values), $f'(1) = 0$. So $\boxed{\text{YES}}$. □