4.3.45 Consider $g(x) = x(\ln x)^2$. Where is f INC/DEC? What are extrema?

Solution

We have $g'(x) = [1](\ln x)^2 + (x)\left[2\ln x\left[\frac{1}{x}\right]\right]$
$= (\ln x)^2 + 2\ln x = \ln x(\ln x + 2)$.

So we have a critical point when either $\ln x = 0$ (that is, when $x = 1$) or $\ln x + 2 = 0$ or $\ln x = -2$ or $e^{\ln x} = e^{-2}$ or $x = e^{-2}$. We perform a SIGN TEST on g':

	$(0, e^{-2})$	$(e^{-2}, 1)$	$(1, \infty)$
TEST VALUE b	e^{-3}	e^{-1}	e
$g'(b)$	$\ln(e^{-3}) \times (\ln(e^{-3})+2) = -3(-3+2) = 3$	$\ln(e^{-1}) \times (\ln(e^{-1})+2) = -(-1+2) = -1$	$\ln(e)(\ln(e)+2) = 1(1+2) = 3$
$g'(x)$	$+$	$-$	$+$
$g(x)$	INC	DEC	INC

$\boxed{\text{So } g \text{ is INC on } (0, e^{-2}] \cup [1, \infty) \text{ and } g \text{ is DEC on } [e^{-2}, 1].}$

By The First Derivative Test for Local Extrema (Theorem 4.3.A), g has a local MAX at $x = e^{-2}$ of $g(e^{-2}) = e^{-2}(\ln e^{-2})^2 = e^{-2}(-2)^2 = 4e^{-2}$ and g has a local MIN at $x = 1$ of $g(1) = (1)(\ln(1))^2 = 0$.

Since the domain of $g(x) = x(\ln x)^2$ is $(0, \infty)$ then $g(x) \geq 0$ for all $x \in (0, \infty)$ and so g has an absolute MIN at $x = 1$ of $g(1) = 0$. Now $g(x) = x(\ln x)^2$ can be made arbitrarily large by making x sufficiently large, so g has no absolute MAX.

$\square$