4.4.41 Consider $y = f(x) = \dfrac{x^2-3}{x-2}$. Identify any local and absolute extreme points and inflection points. Graph the function and include any asymptotes.

Solution

First the domain of f is $(-\infty, 2) \cup (2, \infty)$.

Now
$$f'(x) = \frac{[2x](x-2) - (x^2-3)[1]}{(x-2)^2}$$
$$= \frac{x^2-4x+3}{(x-2)^2} = \frac{(x-1)(x-3)}{(x-2)^2}.$$

So $x = 1$ and $x = 3$ are critical points since $f' = 0$ there. So we perform a sign test on f':

	$(-\infty, 1)$	$(1, 2)$	$(2, 3)$	$(3, \infty)$
TEST VALUE k	0	$3/2$	$5/2$	4
$f'(k)$	$\frac{((0)-1)((0)-3)}{((0)-2)^2} = 3/4$	$\frac{(\frac{3}{2})-1)((\frac{3}{2})-3)}{((\frac{3}{2})-2))^2} = \frac{(\frac{1}{2})(\frac{-3}{2})}{(-1/2)^2}$	$\frac{((\frac{5}{2})-1)((\frac{5}{2})-3)}{((\frac{5}{2})-2)^2} = \frac{(3/2)(-1/2)}{(1/2)^2}$	$\frac{((4)-1)((4)-3)}{((4)-2)^2} = \frac{(3)(1)}{(2)^2}$
$f'(x)$	+	−	−	+
$f(x)$	INC	DEC	DEC	INC

Notice that f is INC on $(-\infty, 1] \cup [3, \infty)$ and f is DEC on $[1, 2) \cup (2, 3]$.

By the First Derivative for Local Extrema (Theorem 4.3.A) f has a local MAX at $x = 1$ of $f(1) = \frac{(1)^2-3}{(1)-2} = 2$ and f has a

$\boxed{\text{local MIN at } x = 3 \text{ of } f(3) = \frac{(3)^2 - 3}{(3) - 2} = 6.}$

next, $f''(x) = \frac{d}{dx}\left[\frac{x^2 - 4x + 3}{(x-2)^2}\right]$

$$= \frac{[2x-4](x-2)^2 - (x^2-4x+3)[2(x-2)[1]]}{((x-2)^2)^2}$$

$$= \frac{2(x-2)((x-2)(x-2) - (x^2-4x+3))}{(x-2)^4}$$

$$= \frac{2(x^2-4x+4-x^2+4x-3)}{(x-2)^3} = \frac{2}{(x-2)^3}.$$

So there are $\boxed{\text{no potential points of inflection}}$ (notice 2 is not in the domain of f).

We perform a sign test on f'':

	$(-\infty, 2)$	$(2, \infty)$
TEST VALUE k	0	3
$f''(k)$	$2/((0)-2)^3 = -1/4$	$2/((3)-2)^3 = 2$
$f''(x)$	$-$	$+$
$f(x)$	CD	CU

Notice that $f(x) = \frac{x^2-3}{x-2}$ has a vertical asymptote at $x = 2$ by Dr. Bob's Infinite Limits Theorem. So we $\lim_{x \to 2^-} f(x) = \pm\infty$ and $\lim_{x \to 2^+} f(x) = \pm\infty$.

So we consider a SIGN DIAGRAM to determine these limits.

For $x \to 2^-$ we have $\frac{x^2-3}{x-2} \Rightarrow \frac{(+)}{(-)} = -$ and so $\lim_{x \to 2^-} f(x) = -\infty$. For $x \to 2^+$ we have $\frac{x^2-3}{x-2} \Rightarrow \frac{(+)}{(+)} = +$ and so $\lim_{x \to 2^+} f(x) = +\infty$.

Since $f(x) = \frac{x^2-3}{x-2}$ is a rational function with the degree of the numerator one greater than the degree of the denominator then (by definition) $y = f(x)$ has an oblique asymptote. We consider

$$
\begin{array}{r}
x + 2 \\
(x-2)\overline{\big) x^2 \qquad -3} \\
\underline{-(x^2-2x)} \\
2x - 3 \\
\underline{-(2x-4)} \\
1
\end{array}
$$

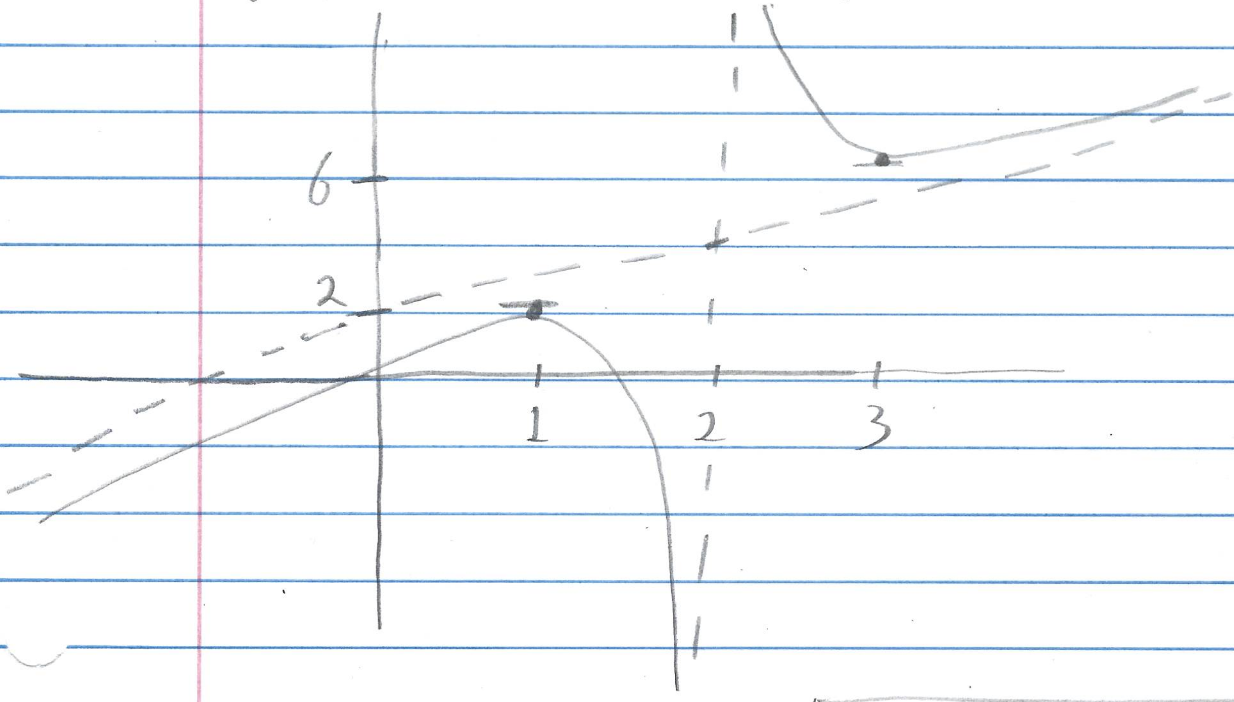
So $\frac{x^2-3}{x-2} = (x+2) + \frac{1}{x-2}$.

Now $\lim_{x \to \pm\infty} \left(\frac{1}{x-2}\right) = \lim_{x \to \pm\infty} \left(\frac{1}{x-2}\right)\left(\frac{1/x}{1/x}\right)$

$= \lim_{x \to \pm\infty} \frac{1/x}{x/x - 2/x} = \lim_{x \to \pm\infty} \left(\frac{1/x}{1-2/x}\right)$

$= \frac{\lim_{x \to \pm\infty} (1/x)}{1 - 2\lim_{x \to \pm\infty} (1/x)} = \frac{0}{1-2(0)} = 0$, ...

So $y = x+2$ is an oblique asymptote of $y = f(x)$. The graph is



Notice that f has $\boxed{\text{no absolute extrema.}}$

$\square$