4.4.97 Consider $y = f(x) = \dfrac{x^3 - 3x^2 + 3x - 1}{x^2 + x - 2}$. Apply the seven steps in the "Procedure for Graphing."

Solution

① First $f(x) = \dfrac{(x-1)^3}{(x+2)(x-1)} = \dfrac{(x-1)^2}{x+2}$ if $x \neq 1$.

So the domain of f is $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$. Since $f(x) \neq f(-x)$ then f is not symmetric with respect to the y-axis. Since $f(x) \neq -f(-x)$ then f is not symmetric with respect to the origin. Notice

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{(x-1)^3}{(x+2)(x-1)} = \lim_{x \to 1} \frac{(x-1)^2}{(x+2)}$$

$$= \frac{((1)-1)^2}{(1)+2} = \frac{0}{3} = 0.$$

② We have

$$y' = f'(x) = \frac{[2(x-1)](x+2) - (x-1)^2[1]}{(x+2)^2}$$

$$= \frac{(x-1)\,(2x+4-(x-1))}{(x+2)^2} = \frac{(x-1)(x+5)}{(x+2)^2}.$$

③ So $x = -5$ is a critical point since $f'(-5) = 0$. Notice that neither $x = 1$ nor $x = -2$ are in the domain of f and so are not critical points.

4 We consider

	$(-\infty, -5)$	$(-5, -2)$	$(-2, 1)$
TEST VALUE k	-6	-3	0
$f'(k)$	$\frac{((-6)-1)((-6)+5)}{((-6)+2)^2}$ $= (-7)(-1)/(-4)^2$	$\frac{((-3)-1)((-3)+5)}{((-3)+2)^2}$ $= (-4)(2)/(-1)^2$	$\frac{((0)-1)((0)+5)}{((0)+2)^2}$ $(-1)(5)/(2)^2$
$f'(x)$	$+$	$-$	$-$
$f(x)$	INC	DEC	DEC

	$(1, \infty)$
TEST VALUE k	2
$f'(k)$	$\frac{((2)-1)((2)+5)}{((2)+2)^2} = \frac{(1)(7)}{(4)^2}$
$f'(x)$	$+$
$f(x)$	INC

So f is INC on $(-\infty, -5] \cup (1, \infty)$.
f is DEC on $[-5, -2) \cup (-2, 1)$.

5 Since $f'(x) = \dfrac{x^2+4x-5}{(x+2)^2}$ then

$$f''(x) = \frac{[2x+4](x+2)^2 - (x^2+4x-5)[2(x+2)]}{((x+2)^2)^2}$$

$$= \frac{(2x+4)((x+2)^2 - (x^2+4x-5))}{(x+2)^4}$$

$$= \frac{2(x+2)(x^2+4x+4-x^2-4x+5)}{(x+2)^4} = \frac{2(9)}{(x+2)^3}.$$

Since $f''(x) = \dfrac{18}{(x+2)^3}$, then there are no critical points (since -2 is not in the domain of f). So we consider

	$(-\infty, -2)$	$(-2, 1)$	$(1, \infty)$
TEST VALUE k	-3	0	2
$f''(k)$	$18/((-3)+2)^3 = -18$	$18/((0)+2)^3 = 9/4$	$18/((2)+2)^3 = 9/32$
$f''(x)$	$-$	$+$	$+$
$f(x)$	CD	CU	CU

So f is CU on $(-2, 1)$ and CD on $(-\infty, -2) \cup (1, \infty)$.

(6) Since $f(x) = \dfrac{(x-1)^2}{x+2}$ if $x \neq 1$, then by

Dr. Bob's Infinite Limits Theorem, $y = f(x)$ has a vertical asymptote at $x = -2$, $\lim\limits_{x \to -2^-} f(x) = \pm\infty$, and $\lim\limits_{x \to -2^+} f(x) = \pm\infty$.

For $x \to -2^-$, $\dfrac{(x-1)^2}{x+2} \Rightarrow \dfrac{(+)}{(-)} = -$ and so

$\lim\limits_{x \to -2^-} f(x) = -\infty$. For $x \to -2^+$,

$\dfrac{(x-1)^2}{x+2} \Rightarrow \dfrac{(+)}{(+)} = +$ and so $\lim\limits_{x \to -2^+} f(x) = \infty$.

Since $f(x) = \dfrac{(x-1)^2}{x+2} = \dfrac{x^2 - 2x + 1}{x+2}$ if $x \neq 1$

is a rational function and the degree of the numerator is one more than the degree of the denominator then $y = f(x)$ has (by definition) an oblique asymptote. We have

$$\begin{array}{r} x - 4 \\ (x+2)\overline{)\,x^2 - 2x + 1} \\ -(x^2 + 2x) \\ \hline -4x + 1 \\ -(-4x - 8) \\ \hline 9 \end{array}$$

so that

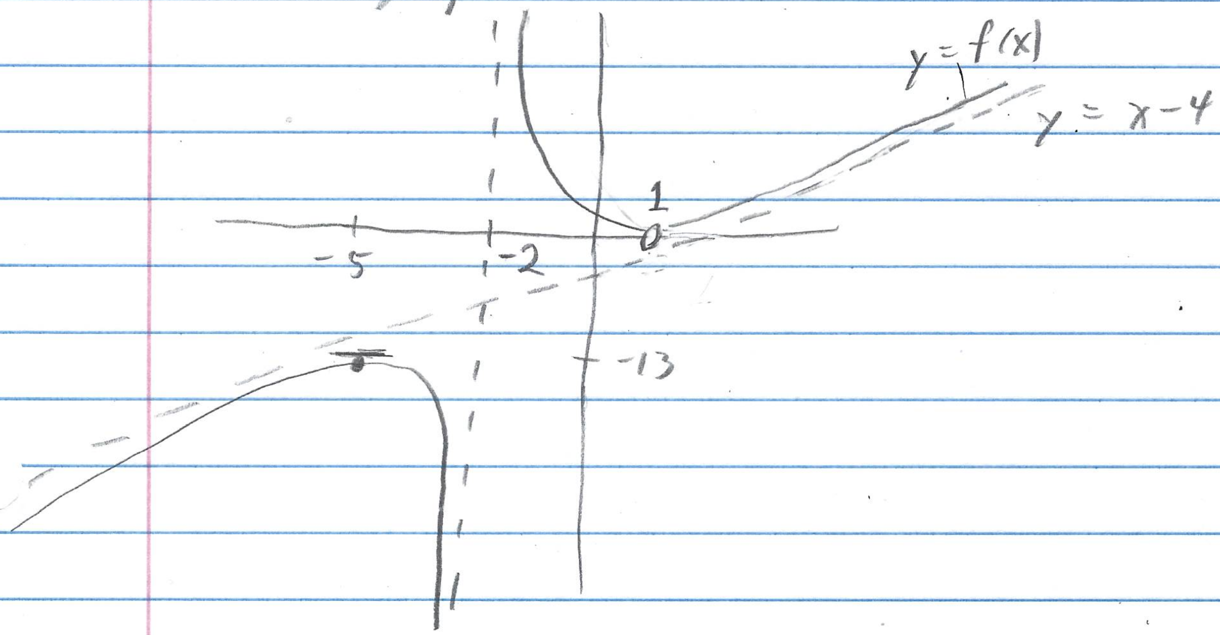
$$f(x) = x - 4 + \frac{9}{x+2},$$

and hence $y = x - 4$ is an oblique asymptote.

(7) Notice that $f(0) = \frac{((0)-1)^2}{(0)+2} = \frac{1}{2}$ (this is the y-intercept). By the First Derivative Test for local Extrema (Stewart 4.3.1), f has a local MAX at $x = -5$ of

$$f(-5) = \frac{((-5)-1)^2}{(-5)+2} = \frac{36}{-3} = -12$$

So the graph is:



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