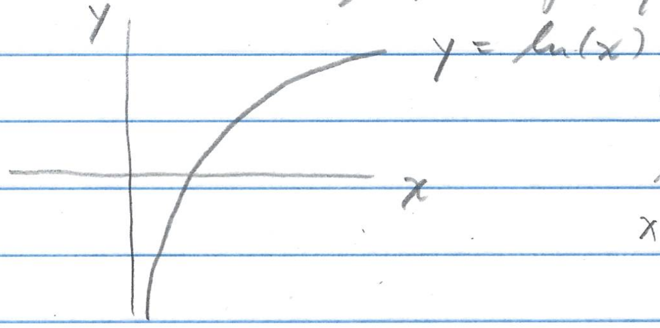
4.5.39 Evaluate $\lim_{x \to 0^+} \frac{(\ln x)^2}{\ln(\sin x)}$.

Solution

Recall the graph of $y = \ln(x)$:



So, $\lim_{x \to 0^+} \ln(x) = -\infty$.

Notice then that $\lim_{x \to 0^+} (\ln x)^2 = \infty$

and $\lim_{x \to 0^+} \ln(\sin x) = -\infty$ (right?).

So, the given limit is of the ∞/∞ indeterminate form. So by l'Hopital's Rule (Theorem 4.5.A) we have

$$\lim_{x \to 0^+} \frac{(\ln x)^2}{\ln(\sin x)} \overset{\infty/\infty}{=\!=} \lim_{x \to 0^+} \frac{2(\ln x)\left[\frac{1}{x}\right]}{\frac{1}{\sin(x)}[\cos(x)]}$$

$$= 2 \lim_{x \to 0^+} \frac{(\sin x)(\ln x)}{x(\cos x)} = 2\left(\lim_{x \to 0^+} \frac{\sin x}{x}\right)\left(\lim_{x \to 0^+} \frac{\ln x}{\cos x}\right)$$

$$= 2(1) \frac{\lim_{x \to 0^+} (\ln x)}{\lim_{x \to 0^+} (\cos x)}$$ since $\lim_{x \to 0} \left(\frac{\sin x}{x}\right) = 1$

$$= \frac{2}{(1)} \lim_{x \to 0^+} (\ln x) \quad \text{since } \lim_{x \to 0} \cos(x) = \cos(0) = 1$$

$$= \boxed{-\infty} \quad \text{since } \lim_{x \to 0^+} (\ln x) = -\infty. \quad \square$$