4.5.53 Evaluate $\lim_{x \to \infty} (\ln x)^{1/x}$.

Solution

Notice that $\lim_{x \to \infty} \ln(x) = \infty$

and $\lim_{x \to \infty} \left(\frac{1}{x}\right) = 0$, so the given limit

is of the ∞^0 indeterminant form!

We consider

$$\ln\left(\lim_{x \to \infty} (\ln x)^{1/x}\right) = \lim_{x \to \infty} \ln\left((\ln x)^{1/x}\right)$$

(since ln is continuous)

$$= \lim_{x \to \infty} \left(\frac{1}{x} \ln(\ln x)\right) = \lim_{x \to \infty} \frac{\ln(\ln x)}{x}$$

$$\overset{\infty/\infty}{=} \lim_{x \to \infty} \frac{\frac{1}{\ln x} \left[\frac{1}{x}\right]}{1} = \lim_{x \to \infty} \left(\frac{1}{x \ln x}\right)$$

$$= \left(\lim_{x \to \infty} \frac{1}{x}\right)\left(\lim_{x \to \infty} \frac{1}{\ln x}\right)$$

$$= (0)(0) \quad \text{since } \lim_{x \to \infty} (\ln x) = \infty$$

$$= 0.$$

Finally, we have

$$e^{\ln\left(\lim_{x\to\infty}(\ln x)^{1/x}\right)} = e^0$$

or $\boxed{\lim_{x\to\infty}(\ln x)^{1/x} = 1}$. $\square$