4.5.61 Evaluate $\lim_{x\to\infty}\left(\frac{x+2}{x-1}\right)^x$.

Solution

Well, notice $\lim_{x\to\infty}\left(\frac{x+2}{x-1}\right) \overset{\infty/\infty}{=} \lim_{x\to\infty}\frac{1}{1} = 1$

and, of course, $\lim_{x\to\infty} x = \infty$ so the given limit is of the 1^∞ indeterminant form.

So, we consider $y = \left(\frac{x+2}{x-1}\right)^x$ and

$$\ln(y) = \ln\left(\frac{x+2}{x-1}\right)^x = x\ln\left(\frac{x+2}{x-1}\right).$$

Next,

$$\lim_{x\to\infty}(\ln(y)) = \lim_{x\to\infty} x\ln\left(\frac{x+2}{x-1}\right)$$ ($0\cdot\infty$ indeterminant form)

$$= \lim_{x\to\infty}\frac{\ln\left(\frac{x+2}{x-1}\right)}{1/x}$$

$$\overset{0/0}{=} \lim_{x\to\infty}\frac{\frac{1}{\left(\frac{x+2}{x-1}\right)}\left[\frac{[1](x-1)-(x+2)[1]}{(x-1)^2}\right]}{-1/x^2}$$

$$= \lim_{x\to\infty}\left(\frac{x-1}{x+2}\right)\left(\frac{-3}{(x-1)^2}\right)(-x^2)$$

$$= \lim_{x\to\infty}\frac{3x^2(x-1)}{(x+2)(x-1)^2} = \lim_{x\to\infty}\frac{3x^2}{(x+2)(x-1)}$$

$$= \lim_{x\to\infty} \frac{3x^2}{x^2+x-2} \overset{\infty/\infty}{=\!=} \lim_{x\to\infty}\left(\frac{6x}{2x+2}\right)$$

$$\overset{\infty/\infty}{=\!=} \lim_{x\to\infty}\left(\frac{6}{2}\right) = \lim_{x\to\infty}(3) = 3.$$

So $\lim_{x\to\infty}(\ln(y)) = 3$ or

$$e^{\lim_{x\to\infty}(\ln y)} = e^3$$

or $e^{\ln\left(\lim_{x\to\infty} y\right)} = e^3$ since $\ln(y)$ is continuous

$$\lim_{x\to\infty} y = e^3$$

or $\lim_{x\to\infty}\left(\frac{x+2}{x-1}\right)^x = \boxed{e^3}$. □