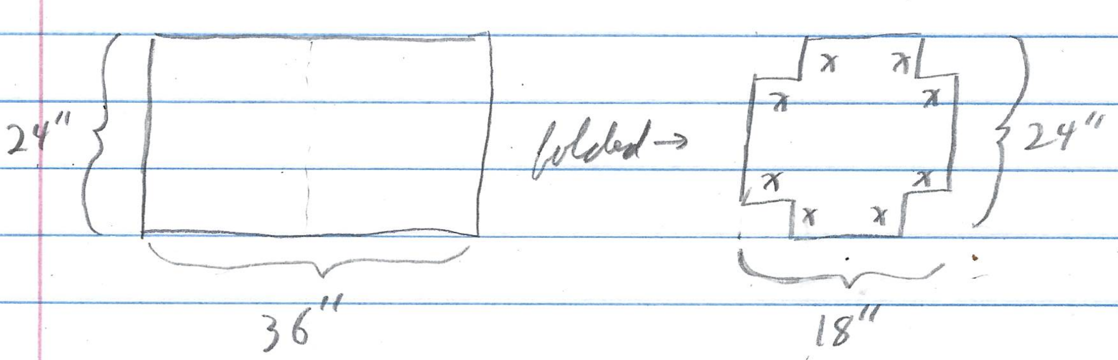
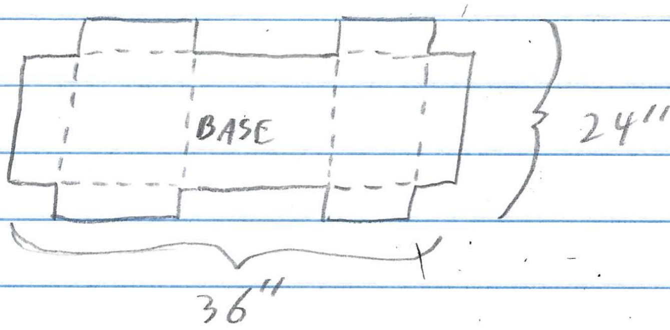
4.6.17 A 24-in.-by-36 in. sheet of cardboard is folded in half to form a 24-in.-by-18 in. rectangle as shown below. Then four congruent squares of side length x are cut from the corners of the folded rectangle. The sheet is unfolded, and the six tabs are folded up to form a box with sides and a lid. Find the maximum volume of the box.

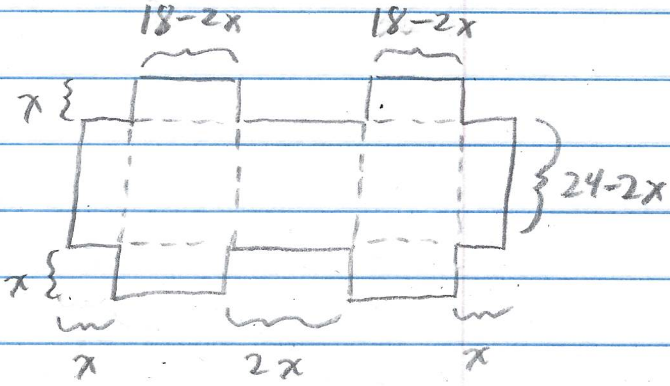


unfolded →

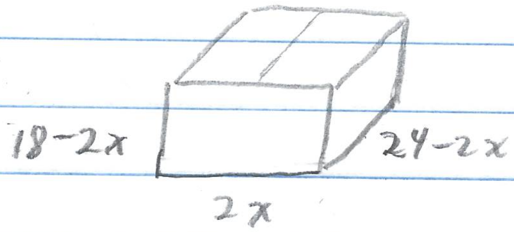


Solution

(2) and (3) We add additional information to the last picture:



The box then has dimensions:



So the volume of the box as a function of x is $V(x) = 2x(18-2x)(24-2x)$.

(4) The question is: What is the MAX of $V(x)$ for $x \in [0, 9]$?

(5) We have

$$V'(x) = [2](18-2x)(24-2x) + (2x)[-2](24-2x) + (2x)(18-2x)[-2]$$
$$= 2\left((18)(24) - 48x - 36x + 4x^2\right) - 4x(24-2x) - 4x(18-2x)$$
$$= (36)(24) - 168x + 8x^2 - 4x(42-4x)$$
$$= 8x^2 + 16x^2 - 168x - 168x + (36)(24)$$
$$= 24x^2 - 336x - (36)(24)$$
$$= 24(x^2 - 14x + 36).$$

Now we set $V'(x) = 0$ which implies $x^2 - 14x + 36 = 0$ or

$$x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(36)}}{2(1)} = \frac{14 \pm \sqrt{52}}{2} = 7 \pm \sqrt{13}.$$

Notice that $7+\sqrt{13} > 9$, so the only critical point in $[0,9]$ is $7-\sqrt{13}$.

At the endpoints $x=0$ and $x=9$, we have $V(x)=0$. At $x=7-\sqrt{13}$ we have

$$V(7-\sqrt{13}) = 2(7-\sqrt{13})(18-2(7-\sqrt{13}))(24-2(7-\sqrt{13}))$$

$$= 2(7-\sqrt{13})\ (4+2\sqrt{13})\ (10+2\sqrt{13})$$

$$= (14-2\sqrt{13})(40+28\sqrt{13}+52) = (14-2\sqrt{13})(92+28\sqrt{13})$$

$$= (14)(92) + (14)(28\sqrt{13}) - 184\sqrt{13} - (56)(13)$$

$$= 1288 + 392\sqrt{13} - 184\sqrt{13} - 728$$

$$= 560 + 208\sqrt{13} \text{ in}^3$$

$$\approx 1310 \text{ in}^3.$$

$\boxed{\text{So the MAX volume is } 560+208\sqrt{13} \text{ in}^3 \text{ and occurs when } x = 7-\sqrt{13} \text{ in.}}$ $\square$