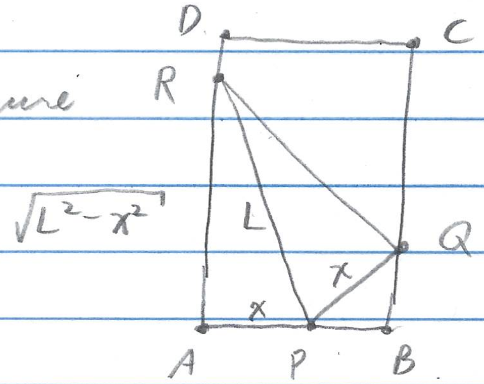
4.6.25 A rectangular sheet of 8.5-in.-by-11-in. paper is placed on a flat surface. One of the corners is placed on the opposite longer edge, as shown in the figure, and held there as the paper is smoothed flat. The problem is to make the length of the crease as small as possible. Call the length L.

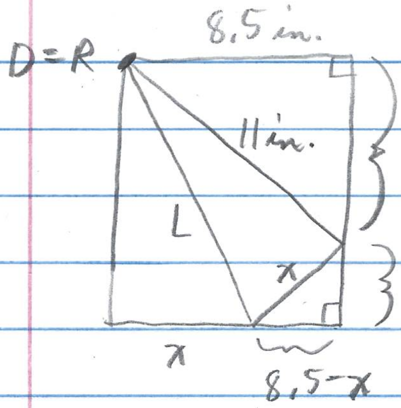
(a) Show that $L^2 = 2x^3/(2x-8.5)$.

Solution

We have the figure



Notice that we could simply fold the paper in half vertically and we would have the corner of the paper at point A folded over to point B, in which case the crease has length $L = 11$ in. and $x = 8.5/2 = 4.25$ in. But the figure is meant to imply that the point labeled R is on the right hand side of the sheet of paper. This implies a lower bound of x which is greater than 4.25 in. Consider the case when points R and D coincide...



By the Pythagorean Theorem

$\sqrt{11^2-8.5^2} = \sqrt{48.75}$

$11-\sqrt{48.75}$

Applying the Pythagorean Theorem to the lower right we have

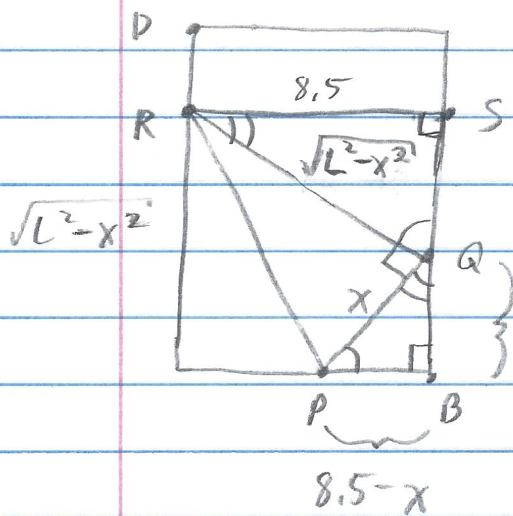
$$x^2 = (8.5-x)^2 + (11-\sqrt{48.75})^2$$
$$= (8.5)^2 - 17x + x^2 + (11-\sqrt{48.75})^2$$

or $17x = (8.5)^2 + (11-\sqrt{48.75})^2$

or $$x = \frac{(8.5)^2 + (11-\sqrt{48.75})^2}{17} = x^* \approx 5.2 \text{ in}$$

So for the given figure to hold, x must be at least x^* (for which the length of the crease is $L = \sqrt{11^2 + (x^*)^2} \approx 12.2$ in).

So we can consider two similar triangles in the given figure:



With the equal angles as indicated, we have that triangles RSQ and QBP are similar.

$\sqrt{x^2-(8.5-x)^2} = \sqrt{x^2-(8.5)^2+17x-x^2}$

$= \sqrt{17x-(8.5)^2}$

By the Pythagorean Theorem

Since RSQ and QBP are similar triangles then the lengths of corresponding sides are in the same ratio. So

$$\frac{\text{length of } RQ}{\text{length of } RS} = \frac{\text{length of } QP}{\text{length of } QB} \quad \text{or}$$

$$\frac{\sqrt{L^2 - x^2}}{8.5} = \frac{x}{\sqrt{17x - (8.5)^2}}, \text{ so}$$

$$\sqrt{L^2 - x^2} = \frac{8.5x}{\sqrt{17x - (8.5)^2}} \quad \text{or} \quad L^2 - x^2 = \frac{(8.5)^2 x^2}{17x - (8.5)^2}$$

$$\text{or } L^2 - x^2 = \frac{8.5x^2}{2x - 8.5} \quad \text{or } L^2 = \frac{8.5x^2}{2x - 8.5} + x^2$$

$$\text{or } L^2 = \frac{8.5x^2}{2x - 8.5} + \frac{x^2(2x - 8.5)}{2x - 8.5} = \frac{2x^3}{2x - 8.5},$$

as claimed. Notice that this formula is valid for the given figure and this holds for $x^* = \dfrac{(8.5)^2 + (11 - \sqrt{48.75})^2}{17} \approx 5.2$ (as discussed above) to $x = 8.5$ in.

(b) What value of x minimizes L^2?

<u>Solution</u>

Since $L^2 = \dfrac{2x^3}{2x - 8.5} = \ell(x)$, we minimize ℓ on $[x^*, 8.5]$ where $x^* \approx 5.2$.

We have $\ell'(x) = \dfrac{[6x^2](2x-8.5)-(2x^3)[2]}{(2x-8.5)^2}$

$$= \frac{12x^3 - 51x^2 - 4x^3}{(2x-8.5)^2} = \frac{8x^3 - 51x^2}{(2x-8.5)^2}$$

$$= \frac{x^2(8x-51)}{(2x-8.5)^2}.$$

So the only critical point of ℓ in $[x^*, 8.5]$ is $x = 51/8$ in. From above we had $L \approx 12.2$ in. when $x = x^*$ (so $L^2 \approx 148$).

We then consider $\ell(x) = L^2$ at the endpoints and the critical point:

x	x^*	$51/8$	8.5
$L^2 = \ell(x)$	≈ 148	$7803/64$	$289/2$

since $$\ell\left(\tfrac{51}{8}\right) = \frac{2(51/8)^3}{2(51/8)-8.5} = \frac{2(51)^3}{8^3(2(51/8)-8.5)}$$

$$= \frac{2(51)^3}{128(51)-(8^3)(8.5)} = \frac{265{,}302}{2176} = \frac{7803}{64} \approx 122.$$

and $$\ell(8.5) = \frac{2(8.5)^3}{2(8.5)-8.5} = \frac{2(8.5)^3}{8.5} = 2(8.5)^2$$

$$= 144.5 = 289/2.$$

$\boxed{\text{So } L^2 \text{ is minimized when } x = 51/8 \text{ in and that minimum is } 7803/64 \text{ in}^2.}$

(c) What is the minimum value of L?

Solution

We have from part (b) that L^2 has a minimum value of $7803/64$ in^2 so (since the square root function is an increasing function and so it preserves inequalities), then $\boxed{\text{the minimum value of } L \text{ is } \sqrt{7803/64} \text{ in} \approx 11 \text{ in.}}$ $\square$

NOTE: The minimum of L given in part (c) is a little bit bigger than 11 in. An anticlimatic observation is that if we simply folded the sheet of paper in half vertically (in which case points Q and B would coincide, but the given figure would not hold) then the length of the crease would be the smaller value of 11 in.