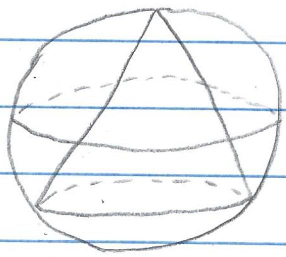
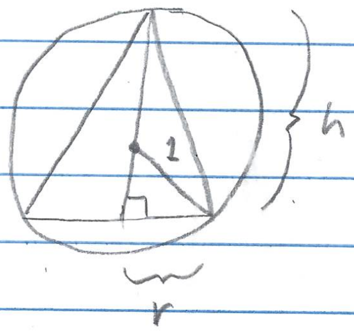
4.6.37 A right circular cone is circumscribed by a sphere of radius 1. Determine the height h and radius r of the cone of maximum volume.

<u>Solution</u>

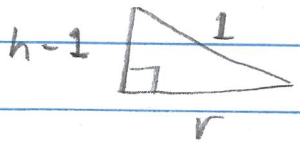
(2) and (3) The picture is:



or in cross-section



Notice that we have the right triangle



so that by the Pythagorean Theorem,

$$r^2 + (h-1)^2 = 1^2$$

$$\text{or } r^2 = 1 - (h-1)^2 = 2h - h^2.$$

Now the volume of a cone is

$$V = \frac{1}{3}\pi r^2 h.$$

(4) SO, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(2h-h^2)h = V(h)$.

(5) The question is to find MAX volume $V(h)$ for $h \in [1, 2]$

Since $V(h) = \frac{1}{3}\pi(2h-h^2)h = \frac{\pi}{3}(2h^2-h^3)$,

so $V'(h) = \frac{\pi}{3}[4h-3h^2] = \frac{\pi h}{3}(4-3h)$.

So V has critical points at $h=0$ and $h=4/3$ where $V'=0$. Notice that only $4/3$ is in $[1,2]$. So consider

h	1	$4/3$	2
$V(h)$	$\frac{\pi}{3}(2(1)^2-(1)^3)$ $=\frac{\pi}{3}$	$\frac{\pi}{3}\left(2\left(\frac{4}{3}\right)^2-\left(\frac{4}{3}\right)^3\right)$ $=\frac{\pi}{3}\left(\frac{32}{9}-\frac{64}{27}\right)$	$\frac{\pi}{3}(2(2)^2-(2)^3)$ $=0$

So V has a MAX of

$$\frac{\pi}{3}\left(\frac{32}{9}-\frac{64}{27}\right) = \frac{\pi}{3}\left(\frac{96}{27}-\frac{64}{27}\right) = \frac{\pi}{3}\frac{32}{27}$$

$$= \frac{32\pi}{81}.$$

This MAX occurs when $h = 4/3$ and ($r^2 = 2h-h^2$ from above) $r = \sqrt{2(4/3)-(4/3)^2}$

$$= \sqrt{\frac{8}{3}-\frac{16}{9}} = \sqrt{\frac{24-16}{9}} = \sqrt{\frac{8}{9}} = \sqrt{\frac{4\cdot 2}{3\cdot 3}} = \frac{2}{3}\sqrt{2}.$$