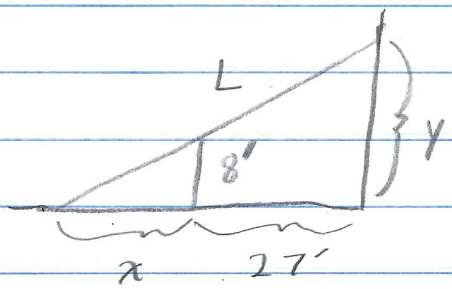
4.6.45 The 8-ft wall shown here stands 27 ft from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.

Solution

(2) and (3) We take the given figure and introduce variables L and x as follows:



Let L be the length of the beam, let x be the distance from the point on the ground at the end of the beam to the wall, and let y be the height of the top of the beam. By similar triangles we have

$$\frac{x}{8} = \frac{x+27}{y} \quad \text{or} \quad y = \frac{8(x+27)}{x} = 8\left(1+\frac{27}{x}\right).$$

By the Pythagorean Theorem,

$$L^2 = (x+27)^2 + y^2.$$

(4) We can now write the length-squared in terms of x as

$$L^2 = \ell(x) = (x+27)^2 + \left(8\left(1+\frac{27}{x}\right)\right)^2.$$

The question is to minimize $L = \sqrt{\ell(x)}$ for $x \in (0, \infty)$.

(5) We first minimize $L^2 = l(x)$ for $x \in (0, \infty)$.
We have

$$l'(x) = 2(x+27) + 2(8(1+27x^{-1}))\left[8(-27x^{-2})\right]$$

$$= 2(x+27) - 128(27)\left(1+\frac{27}{x}\right)\left(\frac{1}{x^2}\right)$$

$$= 2(x+27) - 128(27)\left(\frac{x+27}{x}\right)\left(\frac{1}{x^2}\right)$$

$$= 2(x+27)\left(1 - \frac{64(27)}{x^3}\right)$$

$$= 2(x+27)\left(\frac{x^3 - 64(27)}{x^3}\right).$$

So in $(0, \infty)$ we have a critical point when $x^3 - (64)(27) = 0$ or $x^3 = (64)(27) = 4^3 3^3$ or $x = 12$ ft. We consider the first derivative l' on $(0, \infty)$:

	$(0, 12)$	$(12, \infty)$
TEST VALUE h	1	13
$l'(h)$	$2((1)+27)\left(\frac{1^3 - 12^3}{(1)^3}\right)$	$2((13)+27)\left(\frac{(13)^3 - 12^3}{(13)^3}\right)$
$l'(x)$	$-$	$+$
$l(x)$	DEC	INC

So by the First Derivative Test (Theorem 4.3.A), l has a local minimum at $x = 12$.
Since there is only one critical point of l in $(0, \infty)$, we see that $x = 12$ gives an absolute

minimum of l. Now the square root function is an increasing function so it preserves inequalities. Hence $\sqrt{l(x)} = L$ has an absolute minimum at $x = 12$ ft and this minimum is

$$\sqrt{l(x)} = \left\{((12)+27)^2 + \left(8\left(1+\frac{27}{(12)}\right)\right)^2\right\}^{1/2}$$

$$= \left\{(39)^2 + \left(8\left(\frac{39}{12}\right)\right)^2\right\}^{1/2}$$

$$= \left\{(39)^2 + \left(2\left(\frac{39}{3}\right)\right)^2\right\}^{1/2} = \left\{(39)^2 + (26)^2\right\}^{1/2}$$

$$= \boxed{\sqrt{2197} \text{ ft} \approx 46.87 \text{ ft}} \qquad \square$$