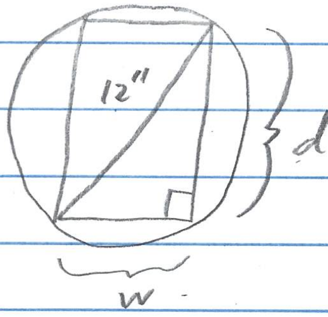
4.6.49(a) The strength S of a rectangular wooden beam is proportional to its width times the square of its depth. Find the dimensions of the strongest beam that can be cut from a 12-in.-diameter cylindrical log.

Solution

(2) and (3) We have the picture (given in the book):



By the Pythagorean Theorem we have $w^2 + d^2 = (12)^2 = 144$. With k as the constant of proportionality, we have that the strength of the beam is

$S = k w d^2$. Since $d^2 = 144 - w^2$ then $S = k w (144 - w^2) = S(w)$.

(4) The question is maximize $S(w) = kw(144 - w^2) = 144kw - kw^3$ for $w \in [0, 12]$.

(5) We have $S'(w) = 144k - 3kw^2 = 3k(48 - w^2)$ so there are critical points of $w = \pm\sqrt{48} = \pm 4\sqrt{3}$ and the only critical point in $[0, 12]$ is $w = 4\sqrt{3}$.

We consider

w	0	$4\sqrt{3}$	12
$S(w)$	$144k(0) - k(0)^3$	$144k(4\sqrt{3}) - k(4\sqrt{3})^3$	$144k(12)$
	$= 0$	$= (576 - 192)k\sqrt{3}$	$-k(12)^3 = 0$
		$= 384k\sqrt{3}$	

$$\boxed{\text{So } S \text{ has a MAX of } 384k\sqrt{3} \text{ when } w = 4\sqrt{3} \text{ in, and } d = \sqrt{144 - (4\sqrt{3})^2} = \sqrt{144 - 48} = \sqrt{96} = 4\sqrt{6} \text{ in.}}$$

□