4.8.105 Solve the initial value problem:

$$\frac{d^2y}{dx^2} = 2-6x \text{ and } y'(0)=4, \quad y(0)=1.$$

Solution

We have $y' = \frac{dy}{dx} \in \int (2-6x)\,dx$

$$= 2x - 6\left(\tfrac{1}{2}x^2\right) + C$$
$$= 2x - 3x^2 + C.$$

So, $y' = 2x - 3x^2 + k_1$ for some constant k_1. To find k_1 we use the initial condition $y'(0) = 4$. This implies

$$y'(0) = 2(0) - 3(0)^2 + k_1 = 4 \text{ or } k_1 = 4.$$

Hence $y'(x) = 2x - 3x^2 + 4$.

Next,

$$y \in \int y'\,dx = \int (2x - 3x^2 + 4)\,dx$$
$$= 2\left(\frac{x^2}{2}\right) - 3\left(\frac{x^3}{3}\right) + 4x + C$$
$$= x^2 - x^3 + 4x + C.$$

So, $y = x^2 - x^3 + 4x + k_2$ for some k_2. Since $y(0) = 1$ we have $y(0) = (0)^2 - (0)^3 + 4(0) + k_2 = 1$ or $k_2 = 1$. Therefore,

$$\boxed{y = x^2 - x^3 + 4x + 1.}$$

□