4.8.41 Evaluate $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}\,dt$ and check your answer by differentiation.

Solution

We have

$$\int \frac{t\sqrt{t}+\sqrt{t}}{t^2}\,dt = \int \frac{t\,t^{1/2}+t^{1/2}}{t^2}\,dt$$

$$= \int \frac{t^{3/2}+t^{1/2}}{t^2}\,dt = \int \frac{t^{3/2}}{t^2} + \frac{t^{1/2}}{t^2}\,dt$$

$$= \int \left(t^{-1/2} + t^{-3/2}\right) dt = \frac{1}{(1/2)}\,t^{1/2} + \frac{1}{(-1/2)}\,t^{-1/2} + C$$

$$= 2t^{1/2} - 2t^{-1/2} + C = \boxed{2\sqrt{t} - \frac{2}{\sqrt{t}} + C}.$$

To check, let $f(t) \in 2\sqrt{t} - \frac{2}{\sqrt{t}} + C$ so that $f(t) = 2\sqrt{t} - \frac{2}{\sqrt{t}} + k = 2t^{1/2} - 2t^{-1/2} + k$ for some constant k. Now

$$f'(t) = \frac{d}{dt}\left[2t^{1/2} - 2t^{-1/2} + k\right] = 2\left(\tfrac{1}{2}t^{-1/2}\right) - 2\left(-\tfrac{1}{2}t^{-3/2}\right) + 0$$

$$= t^{-1/2} + t^{-3/2} = \left(t^{-1/2} + t^{-3/2}\right) t^2/t^2$$

$$= \frac{t^{2-1/2} + t^{2-3/2}}{t^2} = \frac{t^{3/2}+t^{1/2}}{t^2} = \frac{t\sqrt{t}+\sqrt{t}}{t^2},$$

which is the integrand, as needed. □