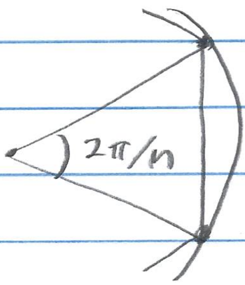
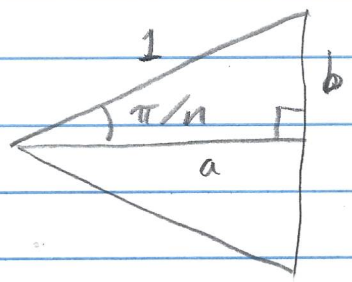
5.1.21 Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of the polygon for the following values of n: (a) 4 (a square), (b) 8 (an octagon), and (c) 16. (d) Compare the areas in parts (a), (b), (c) with the area of the circle.

Solution

For an n-sided regular polygon inscribed in a circle, each side determines a central angle of size $2\pi/n$:



We bisect this angle to find the area of the isosceles triangle determined by this central angle:



As labeled here, $a = \cos\left(\frac{\pi}{n}\right)$ and $b = \sin\left(\frac{\pi}{n}\right)$. So the area of half of the triangle (the right triangle) is $\frac{1}{2}ab = \frac{1}{2}\cos\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)$, and the area of the isosceles triangle is $\cos\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)$. So the area of the n-gon is

$$A_n = n\cos\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{n}\right).$$

(a) With $n=4$, the area of the square is $A_4 = 4\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) = 4\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \boxed{2}$.

(b) With $n=8$, the area of the octagon is

$A_8 = 8\cos\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{8}\right)$. By the half-angle formulas (see Section 1.3),

$$\cos\left(\frac{\pi}{8}\right) = \sqrt{\frac{1+\cos(\pi/4)}{2}} = \sqrt{\frac{1+1/\sqrt{2}}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} \text{ and}$$

$$\sin\left(\frac{\pi}{8}\right) = \sqrt{\frac{1-\cos(\pi/4)}{2}} = \sqrt{\frac{1-1/\sqrt{2}}{2}} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}.$$

So
$$A_8 = 8\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} = 8\,\frac{(\sqrt{2})^2-(1)^2}{2\sqrt{2}}$$
$$= \frac{4}{\sqrt{2}} = \boxed{2\sqrt{2}}.$$

(c) With $n=16$, the area is

$A_{16} = 16\cos\left(\frac{\pi}{16}\right)\sin\left(\frac{\pi}{16}\right)$. We could again use the half-angle formulas to get exact values. Instead, we numerically approximate:

$$A_{16} = 16\cos\left(\frac{\pi}{16}\right)\sin\left(\frac{\pi}{16}\right) \approx 3.061$$

(d) The actual area of the circle is $\pi r^2 = \pi (1)^2 = \pi \approx 3.14159$, so we have (to 3 decimal places):

n	2	4	16	A
area	2	$2\sqrt{2} \approx 2.828$	3.061	3.142

$\square$