5.2.49 For the function $f(x) = 2x^3$, find a formula for the Riemann sum obtained by dividing the interval $[a, b] = [0, 1]$ into n equal subintervals and using the right-hand endpoint for each c_k. Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over $[0, 1]$.

Solution

If we partition the interval $[a, b] = [0, 1]$ into n subintervals of the same width, then that width will be $\Delta x = (b-a)/n = (1-0)/n = 1/n$. The resulting subintervals will be $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$ where $x_k = a + k\,\Delta x = 0 + k(1/n)$ $= k/n$. Using the right-hand endpoint for c_k, we have $c_k = x_k = k/n$. We have the Riemann sum:

$$S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(\frac{1}{n}\right) = \sum_{k=1}^{n} 2\left(\frac{k}{n}\right)^3\left(\frac{1}{n}\right)$$

$$= \frac{2}{n^4}\sum_{k=1}^{n} k^3 = \frac{2}{n^4}\left(\frac{n(n+1)}{2}\right)^2 \quad \text{since} \quad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$= \frac{2}{n^4}\,\frac{n^2(n+1)^2}{4} = \frac{n^2+2n+1}{2n^2}.$$

$$\boxed{\text{The Riemann sum is } \frac{n^2+2n+1}{2n^2}.}$$

Taking a limit as $n \to \infty$ of the Riemann sum gives

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2}$$

$$= \lim_{n\to\infty} \left(\frac{n^2+2n+1}{2n^2}\right)\left(\frac{1/n^2}{1/n^2}\right)$$

$$= \lim_{n\to\infty} \frac{n^2/n^2 + 2n/n^2 + 1/n^2}{2} = \lim_{n\to\infty} \frac{1+2/n+1/n^2}{2}$$

$$= \frac{1 + 2\left(\lim_{n\to\infty} 1/n\right) + \left(\lim_{n\to\infty} 1/n\right)^2}{2}$$

$$= \frac{1+2(0)+(0)^2}{2} = \boxed{\frac{1}{2}}. \quad \square$$