5.3.67 Use a regular partition of $[-1,2]$ and limits to evaluate $\int_{-1}^{2} (3x^2-2x+1)\,dx$.

<u>Solution</u>

Let $f(x) = 3x^2-2x+1$. Then f is continuous on $[-1,2]$ so, by "Integrability of Continuous Functions" (Theorem 5.1), f is integrable on $[-1,2]$. Therefore, we can consider any sequence of partitions which have a norm approaching 0. So we consider an equal-width partition $P = \{x_0, x_1, \ldots, x_n\}$ with $\Delta x_k = \Delta x = (b-a)/n$, $x_k = a + k(b-a)/n$, and $c_k \in [x_{k-1}, x_k]$ satisfies $c_k = x_k = a + k(b-a)/n$. Now $\|P\| = \Delta x = (b-a)/n$, so when $n \to \infty$ we have $\|P\| \to 0$.
So the value of the Riemann integral is given by

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \left(\sum_{k=1}^{n} f(c_k)\,\Delta x_k \right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(-1 + k\left(\frac{3}{n}\right)\right)\left(\frac{3}{n}\right)$$

since $\frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}$

and $c_k = a + k\left(\frac{b-a}{n}\right) = -1 + k\frac{3}{n}$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left(3\left(-1+\frac{3k}{n}\right)^2 - 2\left(-1+\frac{3k}{n}\right)+1\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left(3\left(1-\frac{6k}{n}+\frac{9k^2}{n^2}\right)+2-\frac{6k}{n}+1\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left(6-\frac{24k}{n}+\frac{27k^2}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \left(\sum_{k=1}^{n} 6 - \frac{24}{n}\sum_{k=1}^{n} k + \frac{27}{n^2}\sum_{k=1}^{n} k^2\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \left(6n - \frac{24}{n}\left(\frac{n(n+1)}{2}\right) + \frac{27}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right)$$

since $\sum_{k=1}^{n} c = cn$, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \left(18 - \frac{72}{n}\frac{(n+1)}{2} + \frac{81}{n^2}\frac{(n+1)(2n+1)}{6}\right)$$

$$= 18 - 36\lim_{n\to\infty}\left(\frac{n+1}{n}\right) + \frac{27}{2}\lim_{n\to\infty}\left(\frac{2n^2+3n+1}{n^2}\right)$$

$$= 18 - 36\lim_{n\to\infty}\left(1+\frac{1}{n}\right) + \frac{27}{2}\lim_{n\to\infty}\left(2+3\left(\frac{1}{n}\right)+\frac{1}{n^2}\right)$$

$$= 18 - 36\left(1+\left(\lim_{n\to\infty}\frac{1}{n}\right)\right) + 27 + \frac{81}{6}\left(\lim_{n\to\infty}\frac{1}{n}\right) + \frac{27}{2}\left(\lim_{n\to\infty}\frac{1}{n}\right)^2$$

$$= 18 - 36(1+(0)) + 27 + \frac{81}{6}(0) + \frac{27}{2}(0)^2 = \boxed{9}.$$

$\square$