5.4.36 Guess an antiderivative of the integral and evaluate $\int_1^2 \frac{\ln x}{x}\, dx$.

Solution

Well, the integrand is $f(x) = \frac{\ln x}{x}$.
We don't know an antiderivative of $\ln x$ but we do know that $\frac{d}{dx}[\ln x] = 1/x$.
So let's try $(\ln x)^2$ to see if it is an antiderivative (the derivative of this will have both $\ln x$ and $1/x$ in it):

$$\frac{d}{dx}\left[(\ln x)^2\right] = 2(\ln x)\left[\frac{1}{x}\right] = 2\,\frac{\ln x}{x}.$$

So we see that in fact $F(x) = \frac{1}{2}(\ln x)^2$ is an antiderivative of $f(x) = \frac{\ln x}{x}$.
So by the Fundamental Theorem of Calculus, Part 2

$$\int_1^2 \left(\frac{\ln x}{x}\right) dx = \frac{1}{2}(\ln x)^2\Big|_1^2 = \frac{1}{2}(\ln(2))^2 - \frac{1}{2}(\ln(1))^2$$

$$= \frac{1}{2}(\ln(2))^2 - \frac{1}{2}(0)^2 = \boxed{\frac{1}{2}(\ln(2))^2}. \ \square$$