5.4.49 Find dy/dx for

$$y = \int_{-1}^{x} \frac{t^2}{t^2+4}\,dt - \int_{3}^{x} \frac{t^2}{t^2+4}\,dt.$$

Solution

Notice that $f(t) = \frac{t^2}{t^2+4}$ is continuous on all of $\mathbb{R}$ (and so is continuous on $[a,b]$), so by Sum Rule for Derivatives (Theorem 3.3.E) and the Fundamental Theorem of Calculus, Part 1

$$\frac{dy}{dx} = \frac{d}{dx}\left[\int_{-1}^{x} \frac{t^2}{t^2+4}\,dt - \int_{3}^{x} \frac{t^2}{t^2+4}\,dt\right]$$

$$= \frac{d}{dx}\left[\int_{-1}^{x} \frac{t^2}{t^2+4}\,dt\right] + \frac{d}{dx}\left[-\int_{3}^{x} \frac{t^2}{t^2+4}\,dt\right]$$

$$= \frac{x^2}{x^2+4} - \frac{x^2}{x^2+4} = \boxed{0}.$$

OR: $$\frac{dy}{dx} = \frac{d}{dx}\left[\int_{-1}^{x} \frac{t^2}{t^2+4}\,dt - \int_{3}^{x} \frac{t^2}{t^2+4}\,dx\right]$$

$$= \frac{d}{dx}\left[\int_{-1}^{x} \frac{t^2}{t^2+4}\,dt + \int_{x}^{3} \frac{t^2}{t^2+4}\,dt\right]$$

by Order of Integration (Theorem 5.2(1))

$$= \frac{d}{dx}\left[\int_{-1}^{3} \frac{t^2}{t^2+4}\,dt\right]$$ by Additivity (Theorem 5.2(5))

$= 0$ since $\int_{-1}^{3} \frac{t^2}{t^2+4}\,dt$ is a constant. □