5.6.41 Evaluate $\displaystyle\int_0^1 \frac{4\,ds}{\sqrt{4-s^2}}$.

Solution

Recall that $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C.$

We try to make the given integral look like this one! We have:

$$\int_0^1 \frac{4}{\sqrt{4-s^2}}\,ds = \int_0^1 \frac{4}{\sqrt{4(1-s^2/4)}}\,ds$$

$$= \int_0^1 \frac{4}{2\sqrt{1-(s/2)^2}}\,ds = 2\int_0^1 \frac{1}{\sqrt{1-(s/2)^2}}\,\boxed{ds}$$

let $u = s/2$ and $du = 1/2\,ds$

or $2\,du = ds$

when $s=0$ then $u = 0/2 = 0$ and when $s=1$ $u = 1/2$

$$= 2\int_0^{1/2} \frac{1}{\sqrt{1-u^2}}(2\,du) = 4\int_0^{1/2} \frac{1}{\sqrt{1-u^2}}\,du$$

$$= 4\sin^{-1}(u)\Big|_0^{1/2} = 4\sin^{-1}\left(\tfrac{1}{2}\right) - 4\sin^{-1}(0)$$

$$= 4\left(\frac{\pi}{6}\right) - 4(0) = \boxed{\frac{2\pi}{3}}. \quad \square$$