5.6.85 Find the region enclosed by the curves $x+4y^2=4$ and $x+y^4=1$ for $x \geq 0$.

Solution

Well, $x+4y^2=4$ is equivalent to $x=4-4y^2$. Also $x+y^4=1$ is equivalent to $x=1-y^4$. These intersect when the x-coordinates are the same:

$$4-4y^2 = 1-y^4 \text{ or } y^4-4y^2+3=0$$

or $(y^2-3)(y^2-1)=0$ or

$$(y-\sqrt{3})(y+\sqrt{3})(y-1)(y+1)=0.$$

So these curves intersect when

$y=-\sqrt{3}$, $y=-1$, $y=1$, and $y=\sqrt{3}$.

These correspond to the x-values:

$$y=-\sqrt{3} \Rightarrow x=1-(-\sqrt{3})^4=1-9=-8$$

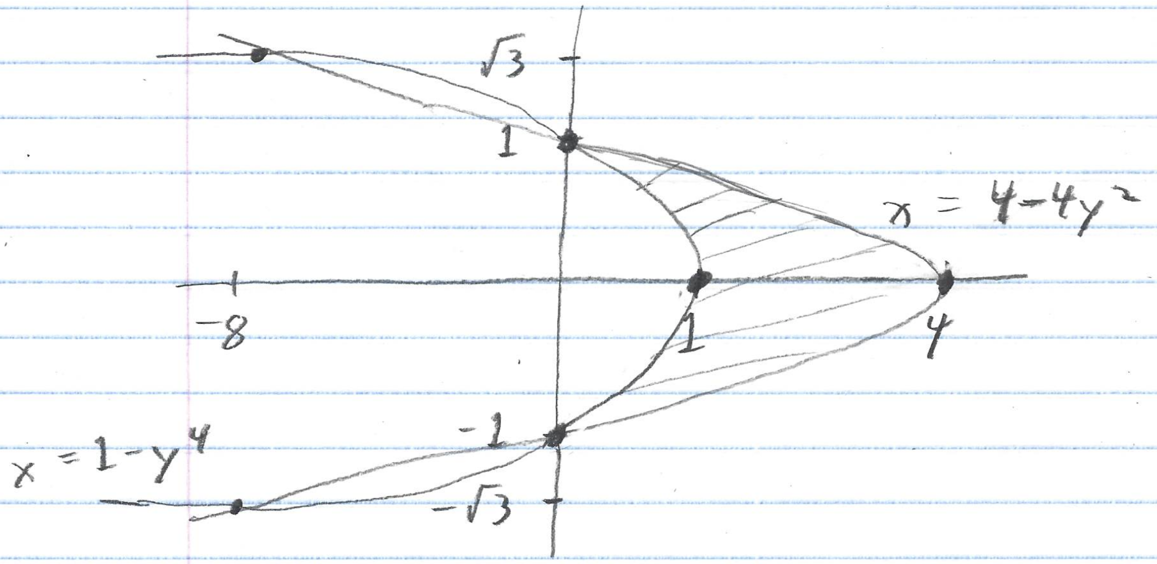
$$y=-1 \Rightarrow x=1-(-1)^4=0$$

$$y=1 \Rightarrow x=1-(1)^4=0$$

$$y=\sqrt{3} \Rightarrow x=1-(\sqrt{3})^4=1-9=-8.$$

We're only interested in $x \geq 0$ (as given).

We need graphs of $x = 4-4y^2$ and $x = 1-y^4$:



The desired region is $A = \int_c^d \left(f(y) - g(y)\right) dy$

or $A = \int_{-1}^{1} \left((4-4y^2) - (1-y^4)\right) dy$

$= \int_{-1}^{1} \left(y^4 - 4y^2 + 3\right) dy$

$= \left(\frac{1}{5} y^5 - \frac{4}{3} y^3 + 3y\right)\Big|_{-1}^{1}$

$= \left(\frac{1}{5}(1)^5 - \frac{4}{3}(1)^3 + 3(1)\right) - \left(\frac{1}{5}(-1)^5 - \frac{4}{3}(-1)^3 + 3(-1)\right)$

$= \left(\frac{1}{5} - \frac{4}{3} + 3\right) - \left(\frac{-1}{5} + \frac{4}{3} - 3\right)$

$= 2\left(\frac{3}{15} - \frac{20}{15} + \frac{45}{15}\right) = 2\left(\frac{28}{15}\right) = \boxed{\frac{56}{15}}$. □