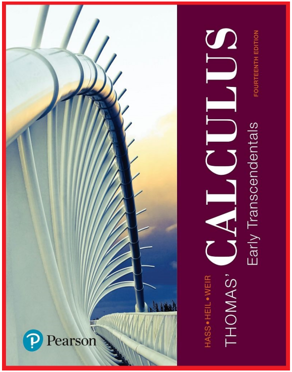
Calculus 1

Appendices

A.1. Real Numbers and the Real Line—Examples and Proofs



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Exercise A.1.6

Exercise A.1.6. Find all $x \in \mathbb{R}$ satisfying $\frac{4}{5}(x-2) < \frac{1}{3}(x-6)$ and show the solution set on the real number line.

Solution. Since $\frac{4}{5}(x-2) < \frac{1}{3}(x-6)$ then, multiplying both sides by 15 and using inequality property (3), we have
$15\left(\frac{4}{5}(x-2)\right) < 15\left(\frac{1}{3}(x-6)\right)$ or (simplifying) $12(x-2) < 5(x-6)$ or (distributing) $12x - 24 < 5x - 30$.

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Solution. Since $\frac{4}{5}(x-2) < \frac{1}{3}(x-6)$ then, multiplying both sides by 15 and using inequality property (3), we have $15\left(\frac{4}{5}(x-2)\right) < 15\left(\frac{1}{3}(x-6)\right)$ or (simplifying) $12(x-2) < 5(x-6)$ or (distributing) $12x - 24 < 5x - 30$. Adding 24 to both sides we have (by inequality property (1)) $(12x-24)+24 < (5x-30)+24$ or (simplifying) $12x < 5x - 6$. Subtracting $5x$ from both sides we have (by inequality property (2)) $(12x) - 5x < (5x-6) - 5x$ or (simplifying) $7x < -6$. Multiplying both sides by $1/7$ we have (by inequality property (3) $(1/7)(7x) < (1/7)(-6)$ or (simplifying) $x < -6/7$.

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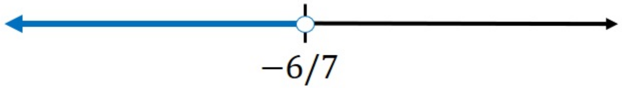
Solution. Since $\frac{4}{5}(x-2) < \frac{1}{3}(x-6)$ then, multiplying both sides by 15 and using inequality property (3), we have
$15\left(\frac{4}{5}(x-2)\right) < 15\left(\frac{1}{3}(x-6)\right)$ or (simplifying) $12(x-2) < 5(x-6)$ or (distributing) $12x - 24 < 5x - 30$. Adding 24 to both sides we have (by inequality property (1)) $(12x-24)+24 < (5x-30)+24$ or (simplifying) $12x < 5x - 6$. Subtracting $5x$ from both sides we have (by inequality property (2)) $(12x) - 5x < (5x-6) - 5x$ or (simplifying) $7x < -6$. Multiplying both sides by $1/7$ we have (by inequality property (3) $(1/7)(7x) < (1/7)(-6)$ or (simplifying) $x < -6/7$.

Exercise A.1.6 (continued)

Exercise A.1.6. Find all $x \in \mathbb{R}$ satisfying $\frac{4}{5}(x-2) < \frac{1}{3}(x-6)$ and show the solution set on the real number line.

Solution (continued). $\ldots x < -6/7$. So the solution set is $\boxed{\{x \in \mathbb{R} \mid x < -6/7\}}$ or the interval $\boxed{(-\infty, -6/7)}$. On the real number line this set is:



$\square$

Exercise A.1.24

Exercise A.1.24. A proof of the Triangle Inequality.
Give the reason justifying each of the numbered steps in the following proof of the Triangle Inequality.

$$
\begin{aligned}
|a+b|^2 &= (a+b)^2 && (1)\\
&= a^2+2ab+b^2 &&\\
&\le a^2+2|a|\,|b|+b^2 && (2)\\
&= |a|^2+2|a|\,|b|+|b|^2 && (3)\\
&= (|a|+|b|)^2 &&\\
|a+b| &\le |a|+|b| && (4)
\end{aligned}
$$

Solution. Since $(a+b)^2 \ge 0$ then $(a+b)^2 = |(a+b)^2|$ by the definition of absolute value. By absolute value property (2), $|(a+b)^2| = |(a+b)(a+b)| = |a+b|\,|a+b| = |a+b|^2$ and so step (1) is justified.

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Solution. Since $(a+b)^2 \geq 0$ then $(a+b)^2 = |(a+b)^2|$ by the definition of absolute value. By absolute value property (2), $|(a+b)^2| = |(a+b)(a+b)| = |a+b|\,|a+b| = |a+b|^2$ and so step (1) is justified.

Exercise A.1.24 (continued 1)

$$
\begin{aligned}
|a+b|^2 &= (a+b)^2 && (1)\\
&= a^2+2ab+b^2 &&\\
&\leq a^2+2|a|\,|b|+b^2 && (2)\\
&= |a|^2+2|a|\,|b|+|b|^2 && (3)\\
&= (|a|+|b|)^2 &&\\
|a+b| &\leq |a|+|b| && (4)
\end{aligned}
$$

Solution (continued). By the definition of absolute value, if $x \geq 0$ then $|x| = x$, and if $x < 0$ (in which case $-x > 0$ by inequality property (4)) then $|x| = -x > 0 > x$; in both cases, $x \leq |x|$. So, with $x = ab$, we have $ab \leq |ab|$ and (by absolute value property (2)) $|ab| = |a|\,|b|$. Hence, $ab \leq |ab| = |a|\,|b|$ and so (by inequality property (3)) $2ab \leq 2|a|\,|b|$. Then (by inequality property (1)) $a^2+b^2+(2ab) \leq a^2+b^2+(2|a|\,|b|)$ and so step (2) is justified.

Exercise A.1.24 (continued 1)

$$\begin{aligned}
|a+b|^2 &= (a+b)^2 && (1)\\
&= a^2+2ab+b^2 \\
&\le a^2+2|a|\,|b|+b^2 && (2)\\
&= |a|^2+2|a|\,|b|+|b|^2 && (3)\\
&= (|a|+|b|)^2 \\
|a+b| &\le |a|+|b| && (4)
\end{aligned}$$

Solution (continued). By the definition of absolute value, if $x \ge 0$ then $|x| = x$, and if $x < 0$ (in which case $-x > 0$ by inequality property (4)) then $|x| = -x > 0 > x$; in both cases, $x \le |x|$. So, with $x = ab$, we have $ab \le |ab|$ and (by absolute value property (2)) $|ab| = |a|\,|b|$. Hence, $ab \le |ab| = |a|\,|b|$ and so (by inequality property (3)) $2ab \le 2|a|\,|b|$. Then (by inequality property (1)) $a^2 + b^2 + (2ab) \le a^2 + b^2 + (2|a|\,|b|)$ and so step (2) is justified.

Exercise A.1.24 (continued 2)

$$\begin{aligned} |a+b|^2 &= (a+b)^2 && (1) \\ &= a^2+2ab+b^2 \\ &\leq a^2+2|a|\,|b|+b^2 && (2) \\ &= |a|^2+2|a|\,|b|+|b|^2 && (3) \\ &= (|a|+|b|)^2 \\ |a+b| &\leq |a|+|b| && (4) \end{aligned}$$

Solution (continued). Since $x^2 \geq 0$ then $x^2 = |x^2|$ by the definition of absolute value. By absolute value property (2), $|x^2| = |xx| = |x|\,|x|$ and so $x^2 = |x|^2$. With $x = a$ we have $a^2 = |a|^2$ and with $x = b$ we have $b^2 = |b|^2$. So $a^2 + 2|a|\,|b| + b^2 = |a|^2 + 2|a|\,|b| + |b|^2$ and step (3) is justified.

Exercise A.1.24 (continued 3)

$$\begin{aligned}
|a+b|^2 &= (a+b)^2 && (1)\\
&= a^2+2ab+b^2\\
&\leq a^2+2|a|\,|b|+b^2 && (2)\\
&= |a|^2+2|a|\,|b|+|b|^2 && (3)\\
&= (|a|+|b|)^2\\
|a+b| &\leq |a|+|b| && (4)
\end{aligned}$$

Solution (continued). Since $|a+b|^2 \leq (|a|+|b|)^2$, then taking square roots of both sides and using the fact that the square root function is an increasing function on non-negative numbers (so it preserves inequalities involving non-negative numbers), we have $\sqrt{(|a+b|)^2} \leq \sqrt{(|a|+|b|)^2}$ or, since $\sqrt{x^2} = |x|$, $||a+b|| \leq ||a|+|b||$. Since $|a+b| \geq 0$ then $||a+b|| = |a+b|$, and since $|a|+|b| \geq 0$ then $||a|+|b|| = |a|+|b|$. Therefore, $|a+b| \leq |a|+|b|$ and step (4) is justified. $\square$

Exercise A.1.24 (continued 3)

$$
\begin{aligned}
|a+b|^2 &= (a+b)^2 && (1)\\
&= a^2 + 2ab + b^2 && \\
&\leq a^2 + 2|a|\,|b| + b^2 && (2)\\
&= |a|^2 + 2|a|\,|b| + |b|^2 && (3)\\
&= (|a|+|b|)^2 && \\
|a+b| &\leq |a| + |b| && (4)
\end{aligned}
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Solution (continued). Since $|a+b|^2 \leq (|a|+|b|)^2$, then taking square roots of both sides and using the fact that the square root function is an increasing function on non-negative numbers (so it preserves inequalities involving non-negative numbers), we have $\sqrt{(|a+b|)^2} \leq \sqrt{(|a|+|b|)^2}$ or, since $\sqrt{x^2} = |x|$, $||a+b|| \leq ||a|+|b||$. Since $|a+b| \geq 0$ then $||a+b|| = |a+b|$, and since $|a|+|b| \geq 0$ then $||a|+|b|| = |a|+|b|$. Therefore, $|a+b| \leq |a|+|b|$ and step (4) is justified. $\square$

Exercise A.1.12

Exercise A.1.12. Express the solution set as an interval or a union of intervals and show the solution set on the real line: $|3y - 7| < 4$.

Solution. By the relationship of intervals to absolute values (property (6)) we have that $|3y - 7| < 4$ is equivalent to $-4 < 3y - 7 < 4$. Adding 7 to each of the three parts (by inequality property (1)) we have $(-4) + 7 < (3y - 7) + 7 < (4) + 7$ or (simplifying) $3 < 3y < 11$.

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$(-4) + 7 < (3y - 7) + 7 < (4) + 7$ or (simplifying) $3 < 3y < 11$.
Multiplying each of the three parts by $1/3$ (by inequality property (3)) we have $3/3 < 3y/3 < 11/3$ or (simplifying) $1 < y < 11/3$. So the solution set $\boxed{\{y \in \mathbb{R} \mid 1 < y < 11/3\}}$ or the interval $\boxed{(1, 11/3)}$. On the real number line this set is:

□

Exercise A.1.16

Exercise A.1.16. Express the solution set as an interval or a union of intervals and show the solution set on the real line: $|1-x|>1$.

Solution. By the relationship of intervals to absolute values (property (7)) we have that $|1-x|>1$ is equivalent to $1-x<-1$ **or** $1-x>1$. Adding x to both sides of each inequality (by inequality property (1)) we have $(1-x)+x<(-1)+x$ **or** $(1-x)+x>(1)+x$, which simplifies to $1<-1+x$ **or** $1>1+x$.

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Exercise A.1.16

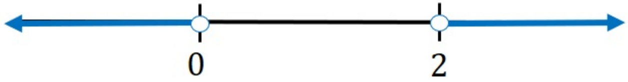
Exercise A.1.16. Express the solution set as an interval or a union of intervals and show the solution set on the real line: $|1 - x| > 1$.

Solution. By the relationship of intervals to absolute values (property (7)) we have that $|1 - x| > 1$ is equivalent to $1 - x < -1$ **or** $1 - x > 1$. Adding x to both sides of each inequality (by inequality property (1)) we have $(1 - x) + x < (-1) + x$ **or** $(1 - x) + x > (1) + x$, which simplifies to $1 < -1 + x$ **or** $1 > 1 + x$. Adding 1 to both sides of the first inequality and subtracting 1 from both sides of the second inequality (by inequality properties (1) and (2)) we have $(1) + 1 < (-1 + x) + 1$ **or** $(1) - 1 > (1 + x) - 1$. This simplifies to the condition on x of $2 < x$ **or** $0 > x$. We have $2 < x$ (or $x > 2$) for $x \in (2, \infty)$. We have $0 > x$ (or $x < 0$) for $x \in (-\infty, 0)$.

Exercise A.1.16 (continued)

Exercise A.1.16. Express the solution set as an interval or a union of intervals and show the solution set on the real line: $|1 - x| > 1$.

Solution. ... We have $0 > x$ (or $x < 0$) for $x \in (-\infty, 0)$. So the solution set is $\boxed{\{x \in \mathbb{R} \mid x < 0\} \cup \{x \in \mathbb{R} \mid x > 2\}}$, or the union of intervals $\boxed{(-\infty, 0) \cup (2, \infty)}$. On the real number line this set is:



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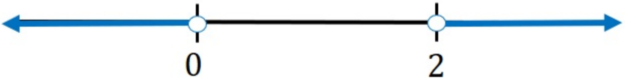

$\square$

Exercise A.1.20

Exercise A.1.20. Solve the inequality $(x-1)^2 < 4$. Express the solution set as an interval or a union of intervals and show them on the real line. Use the result $\sqrt{a^2} = |a|$.

Solution. Since $(x-1)^2 < 4$, then taking square roots of both sides and using the fact that the square root function is an increasing function on non-negative numbers (so it preserves inequalities involving non-negative numbers), we have $\sqrt{(x-1)^2} < \sqrt{4}$ or $|x-1| < 2$.

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