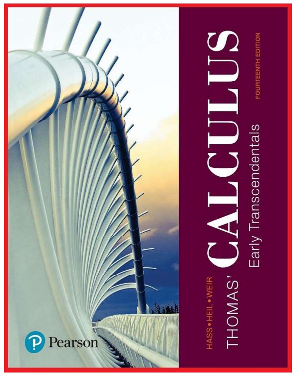
Calculus 1

Appendices

A.2. Mathematical Induction—Examples and Proofs



Example A.2.1

Example A.2.1. Use mathematical induction to prove that for natural number $n \in \mathbb{N}$,

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Prove. First, we check the formula for $n = 1$. This gives $\sum_{i=1}^{1} i = 1 = \frac{(1)((1)+1)}{2} = 1$, which holds. Second, we assume the formula holds for $n = k$, so that we assume

$$1 + 2 + \cdots + k = \sum_{i=1}^{k} i = \frac{(k)((k)+1)}{2}.$$

We want to show that the formula also holds for $n = k + 1$. Consider $1 + 2 + \cdots + k + (k+1) = (1 + 2 + \cdots + k) + (k+1)$.

Example A.2.1 (solution)

Solution (continued). We have

$$\begin{aligned} & (1 + 2 + \cdots + k) + (k+1) \\ = & \left(\sum_{i=1}^{k} i\right) + (k+1) \\ = & \left(\frac{k(k+1)}{2}\right) + (k+1) \text{ by the induction hypothesis} \\ = & \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2} \\ = & \frac{(k+1)((k+1)+1)}{2} = \frac{n(n+1)}{2} \text{ where } n = k+1. \end{aligned}$$

So the formula holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Example A.2.A

Example A.2.A. Prove that for differentiable functions of x, $u_1, u_2, \ldots, u_n$, we have

$$\frac{d}{dx}[u_1 + u_2 + \cdots + u_n] = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}.$$

Proof. First, we check the formula for $n = 1$. This gives $\frac{d}{dx}[u_1] = \frac{du_1}{dx}$, which holds. Notice also that $\frac{d}{dx}[u_1 + u_2] = \frac{du_1}{dx} + \frac{du_2}{dx}$ by the Derivative Sum Rule (Theorem 3.3.E), so that the result also holds for $n = 2$. Second, we assume the formula holds for $n = k$, so that we assume $\frac{d}{dx}[u_1 + u_2 + \cdots + u_k] = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx}$. We want to show that the formula also holds for $n = k + 1$. Consider $\frac{d}{dx}[u_1 + u_2 + \cdots + u_k + u_{k+1}] = \frac{d}{dx}[(u_1 + u_2 + \cdots + u_k) + u_{k+1}]$.

Example A.2.A (continued)

Prove (continued). We have $\frac{d}{dx}[u_1 + u_2 + \cdots + u_k + u_{k+1}]$

$$\begin{aligned} &= \frac{d}{dx}[(u_1 + u_2 + \cdots + u_k) + u_{k+1}] \\ &= \frac{d}{dx}[(u_1 + u_2 + \cdots + u_k)] + \frac{d}{dx}[u_{k+1}] \text{ since the result} \\ &\quad \text{holds for } n = 2 \text{ functions} \\ &= \left(\frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx}\right) + \frac{d}{dx}[u_{k+1}] \\ &\quad \text{by the induction hypothesis} \\ &= \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx} + \frac{du_{k+1}}{dx} \\ &= \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx} \text{ where } n = k+1, \end{aligned}$$

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Exercise A.2.2

Exercise A.2.2. Prove that if $r \neq 1$ then $1 + r + r^2 + \cdots + r^n = \dfrac{1 - r^{n+1}}{1 - r}$ for every natural number $n \in \mathbb{N}$.

Proof. First, we check the formula for $n = 1$. This gives $1 + r = \frac{1-r^2}{1-r} = \frac{(1-r)(1+r)}{1-r} = 1 + r$, which holds. Second, we assume the formula holds for $n = k$, so that we assume $1 + r + r^2 + \cdots + r^k = \dfrac{1 - r^{k+1}}{1 - r}$. We want to show that the formula also holds for $n = k + 1$. Consider $1 + r + r^2 + \cdots + r^k + r^{k+1} = (1 + r + r^2 + \cdots + r^k) + r^{k+1}$.

Exercise A.2.2 (continued)

Exercise A.2.2. Prove that if $r \neq 1$ then $1 + r + r^2 + \cdots + r^n = \dfrac{1 - r^{n+1}}{1 - r}$ for every natural number $n \in \mathbb{N}$.

Proof (continued). We have $1 + r + r^2 + \cdots + r^k + r^{k+1}$

$$\begin{aligned} &= (1 + r + r^2 + \cdots + r^k) + r^{k+1} \\ &= \left(\frac{1 - r^{k+1}}{1 - r}\right) + r^{k+1} \text{ by the induction hypothesis} \\ &= \frac{1 - r^{k+1}}{1 - r} + \frac{r^{k+1}(1 - r)}{1 - r} = \frac{(1 - r^{k+1}) + (r^{k+1} - r^{k+2})}{1 - r} \\ &= \frac{1 - r^{k+2}}{1 - r} = \frac{1 - r^{n+1}}{1 - r} \text{ where } n = k+1, \end{aligned}$$

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Exercise A.2.9

Exercise A.2.9. Sums of Squares
Prove Theorem 5.2.B(2):

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \text{ for all } n \in \mathbb{N}.$$

Proof. First, we check the formula for $n = 1$. This gives $1^2 = \dfrac{(1)((1)+1)(2(1)+1)}{6} = \dfrac{6}{6} = 1$, which holds. Second, we assume the formula holds for $n = k$, so that we assume $1^2 + 2^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$. We want to show that the formula also holds for $n = k + 1$. Consider $1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = (1^2 + 2^2 + \cdots + k^2) + (k+1)^2$.

Exercise A.2.9 (continuous)

Proof (continued). We have $1^2+2^2+\cdots+k^2+(k+1)^2$

$$\begin{aligned}
&= (1^2+2^2+\cdots+k^2)+(k+1)^2\\
&= \left(\frac{k(k+1)(2k+1)}{6}\right)+(k+1)^2 \text{ by the induction hypothesis}\\
&= \frac{k(k+1)(2k+1)}{6}+\frac{6(k+1)^2}{6}=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}\\
&= \frac{(k+1)(k(2k+1)+6(k+1))}{6}=\frac{(k+1)(2k^2+k+6k+6)}{6}\\
&= \frac{(k+1)(2k^2+7k+6)}{6}=\frac{(k+1)(k+2)(2k+3)}{6}\\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}=\frac{n(n+1)(2n+1)}{6} \text{ where } n=k+1,
\end{aligned}$$

so the result holds for $n=k+1$ and, by the mathematical induction principle, the formula holds for all $n\in\mathbb{N}$, as claimed. □

Exercise A.2.10

Exercise A.2.10. Sums of Cubes
Prove Theorem 5.2.B(3):

$$\sum_{i=1}^{n} i^3 = 1^3+2^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2 \text{ for all } n\in\mathbb{N}.$$

Solution. First, we check the formula for $n=1$. This gives $1^3=\left(\frac{(1)((1)+1)}{2}\right)^2=1^2$, which holds. Second, we assume the formula holds for $n=k$, so that we assume $1^3+2^3+\cdots+k^3=\left(\frac{k(k+1)}{2}\right)^2$.
We want to show that the formula also holds for $n=k+1$. Consider $1^3+2^3+\cdots+k^3+(k+1)^3=(1^3+2^3+\cdots+k^3)+(k+1)^3$.

Exercise A.2.10 (continued)

Solution. We have $1^3+2^3+\cdots+k^3+(k+1)^3$

$$\begin{aligned}
&= (1^3+2^3+\cdots+k^3)+(k+1)^3\\
&= \left(\frac{k(k+1)}{2}\right)^2+(k+1)^3 \text{ by the induction hypothesis}\\
&= \left(\frac{k(k+1)}{2}\right)^2+\frac{4(k+1)^3}{4}=\frac{k^2(k+1)^2+4(k+1)^3}{4}\\
&= \frac{(k+1)^2(k^2+4(k+1))}{4}=\frac{(k+1)^2(k^2+4k+4)}{4}\\
&= \frac{(k+1)^2(k+2)^2}{4}=\frac{(k+1)^2((k+1)+1)^2}{4}\\
&= \left(\frac{(k+1)((k+1)+1)}{2}\right)^2=\left(\frac{n(n+1)}{2}\right)^2 \text{ where } n=k+1,
\end{aligned}$$

so the result holds for $n=k+1$ and, by the mathematical induction principle, the formula holds for all $n\in\mathbb{N}$, as claimed. □

Exercise A.2.11

Exercise A.2.11. Prove Theorem 5.2.A, “Algebra for Finite Sums.”

1. *Sum Rule*: $\sum_{i=1}^{n}(a_i+b_i)=\sum_{i=1}^{n}a_i+\sum_{i=1}^{n}b_i$
2. *Difference Rule*: $\sum_{i=1}^{n}(a_i-b_i)=\sum_{i=1}^{n}a_i-\sum_{i=1}^{n}b_i$
3. *Constant Multiple Rule*: $\sum_{i=1}^{n}ca_i=c\sum_{i=1}^{n}a_i$
4. *Constant Value Rule*: $\sum_{i=1}^{n}c=nc$

Exercise A.2.11 (continued 1)

Sum Rule: $\displaystyle\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

Solution. First, we check the formula for $n = 1$. This gives $(a_1 + b_1) = (a_1) + (b_1)$, which holds. Second, we assume the formula holds for $n = k$, so that we assume $\displaystyle\sum_{i=1}^{k}(a_i + b_i) = \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i$. We want to show that the formula also holds for $n = k + 1$. Consider $\displaystyle\sum_{i=1}^{k+1}(a_i + b_i) = \left(\sum_{i=1}^{k}(a_i + b_i)\right) + (a_{k+1} + b_{k+1})$.

Exercise A.2.11 (continued 2)

Solution (continued). We have $\displaystyle\sum_{i=1}^{k+1}(a_i + b_i)$

$$= \left(\sum_{i=1}^{k}(a_i + b_i)\right) + (a_{k+1} + b_{k+1})$$

$$= \left(\sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i\right) + (a_{k+1} + b_{k+1}) \text{ by the induction hypothesis}$$

$$= \left(\sum_{i=1}^{k} a_i + a_{k+1}\right) + \left(\sum_{i=1}^{k} b_i + b_{k+1}\right)$$

by associativity and commutivity of addition in $\mathbb{R}$

$$= \sum_{i=1}^{k+1} a_i + \sum_{i=1}^{k+1} b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \text{ where } n = k+1,$$

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Exercise A.2.11 (continued 3)

Difference Rule: $\displaystyle\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Solution. First, we check the formula for $n = 1$. This gives $(a_1 - b_1) = (a_1) - (b_1)$, which holds. Second, we assume the formula holds for $n = k$, so that we assume $\displaystyle\sum_{i=1}^{k}(a_i - b_i) = \sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i$. We want to show that the formula also holds for $n = k + 1$. Consider $\displaystyle\sum_{i=1}^{k+1}(a_i - b_i) = \left(\sum_{i=1}^{k}(a_i - b_i)\right) + (a_{k+1} - b_{k+1})$.

Exercise A.2.11 (continued 4)

Solution (continued). We have $\displaystyle\sum_{i=1}^{k+1}(a_i - b_i)$

$$= \left(\sum_{i=1}^{k}(a_i - b_i)\right) + (a_{k+1} - b_{k+1})$$

$$= \left(\sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i\right) + (a_{k+1} - b_{k+1}) \text{ by the induction hypothesis}$$

$$= \left(\sum_{i=1}^{k} a_i + a_{k+1}\right) - \left(\sum_{i=1}^{k} b_i + b_{k+1}\right)$$

by associativity and commutivity of addition in $\mathbb{R}$

$$= \sum_{i=1}^{k+1} a_i - \sum_{i=1}^{k+1} b_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \text{ where } n = k+1,$$

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Exercise A.2.11 (continued 5)

$$\textit{Constant Multiple Rule: } \sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

Solution. First, we check the formula for $n = 1$. This gives $(ca_1) = c(a_1)$, which holds. Second, we assume the formula holds for $n = k$, so that we assume $\sum_{i=1}^{k} ca_i = c\sum_{i=1}^{k} a_i$. We want to show that the formula also holds for $n = k+1$. Consider $\sum_{i=1}^{k+1} ca_i = \sum_{i=1}^{k} ca_i + ca_{k+1}$.

Exercise A.2.11 (continued 6)

Solution (continued). We have $\sum_{i=1}^{k+1} ca_i$

$$\begin{aligned} &= \sum_{i=1}^{k} ca_i + ca_{k+1} \\ &= c\sum_{i=1}^{k} a_i + ca_{k+1} \text{ by the induction hypothesis} \\ &= c\left(\sum_{i=1}^{k} a_i + a_{k+1}\right) \text{ since multiplication distributes over addition in } \mathbb{R} \\ &= c\left(\sum_{i=1}^{k+1} a_i\right) = c\sum_{i=1}^{n} a_i \text{ where } n = k+1, \end{aligned}$$

so the result holds for $n = k+1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Exercise A.2.11 (continued 7)

$$\textit{Constant Value Rule: } \sum_{i=1}^{n} c = nc$$

Solution. First, we check the formula for $n = 1$. This gives $(c) = 1c$, which holds. Second, we assume the formula holds for $n = k$, so that we assume $\sum_{i=1}^{k} c = kc$. We want to show that the formula also holds for $n = k+1$. Consider $\sum_{i=1}^{k+1} c = \left(\sum_{i=1}^{k} c\right) + c$.

Exercise A.2.11 (continued 8)

Solution (continued). We have $\sum_{i=1}^{k+1} c$

$$\begin{aligned} &= \left(\sum_{i=1}^{k} c\right) + c \\ &= kc + c \text{ by the induction hypothesis} \\ &= (k+1)c \text{ since multiplication distributes over addition in } \mathbb{R} \\ &= nc \text{ where } n = k+1, \end{aligned}$$

so the result holds for $n = k+1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Example A.2.B. General Product Rule

Example A.2.B. Prove the General Product Rule (see Exercise 3.3.77 for motivation of this result): For differentiable functions $u_1, u_2, \ldots, u_n$, we have that the derivative of the product $u_1 u_2 \cdots u_n$ exists and

$$\begin{aligned}\frac{d}{dx}[(u_1)(u_2)\cdots(u_n)] = \; & [u_1'](u_2)(u_3)\cdots(u_{n-1})(u_n)\\ & +(u_1)[u_2'](u_3)\cdots(u_{n-1})(u_n)\\ & +(u_1)(u_2)[u_3']\cdots(u_{n-1})(u_n)+\cdots\\ & +(u_1)(u_2)(u_3)\cdots[u_{n-1}'](u_n)\\ & +(u_1)(u_2)(u_3)\cdots(u_{n-1})[u_n'].\end{aligned}$$

Example A.2.B (continued 1)

Proof. We introduce a product notation, similar to the summation notation: $u_1 u_2 \cdots u_n = \prod_{i=1}^{n} u_i$. We can then express the claim of this theorem as

$$\frac{d}{dx}\left[\prod_{i=1}^{n} u_i\right] = \sum_{j=1}^{n} u_j' \prod_{i=1, i\neq j}^{n} u_i.$$

First, we check the formula for $n = 1$. This gives $\frac{d}{dx}[u_1] = \sum_{j=1}^{1} u_j' \prod_{i=1,i\neq 1}^{1} u_i = u_1'$, which holds. For clarity, we also check the formula for $n = 2$. This gives $\frac{d}{dx}[u_1 u_2] = \sum_{j=1}^{2} u_j' \prod_{i=1,i\neq j}^{2} u_i = [u_1'](u_2) + (u_1)[u_2']$, which holds by the Derivative Product Rule (Theorem 3.3.G).

Example A.2.B (continued 2)

Proof (continued). Next, we assume the formula holds for $n = k$, so that we assume $\frac{d}{dx}\left[\prod_{i=1}^{k} u_i\right] = \sum_{j=1}^{k} u_j' \prod_{i=1,i\neq j}^{k} u_i$. We want to show that the formula also holds for $n = k + 1$. Consider

$$\frac{d}{dx}\left[\prod_{i=1}^{k+1} u_i\right] = \frac{d}{dx}\left[\left(\prod_{i=1}^{k} u_i\right) u_{k+1}\right]$$

$$= \frac{d}{dx}\left[\prod_{i=1}^{k} u_i\right](u_{k+1}) + \left(\prod_{i=1}^{k} u_i\right)[u_{k+1}'] \text{ by the Derivative Rule for Products (Theorem 3.3.G)}$$

$$= \left[\sum_{j=1}^{k} u_j' \prod_{i=1,i\neq j}^{k} u_i\right](u_{k+1}) + \left(\prod_{i=1}^{k} u_i\right)[u_{k+1}'] \text{ by induction hypothesis}$$

Example A.2.B (continued 3)

Proof (continued). $\ldots \frac{d}{dx}\left[\prod_{i=1}^{k+1} u_i\right]$

$$= \left[\sum_{j=1}^{k} u_j' \prod_{i=1,i\neq j}^{k} u_i\right](u_{k+1}) + \left(\prod_{i=1}^{k} u_i\right)[u_{k+1}']$$

$$= \sum_{j=1}^{k} u_j' \prod_{i=1,i\neq j}^{k+1} u_i + u_{k+1}' \prod_{i=1,i\neq k+1}^{k+1} u_i$$

$$= \sum_{j=1}^{k+1} u_j \prod_{i=1,i\neq j}^{k+1} u_i = \sum_{j=1}^{n} u_j \prod_{i=1,i\neq j}^{n} u_i \text{ where } n = k+1,$$

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □