Appendices
A.2. Mathematical Induction—Examples and Proofs

Example A.2.1

Example A.2.1. Use mathematical induction to prove that for natural number $n \in \mathbb{N}$,

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

**Prove.** First, we check the formula for $n = 1$. This gives

$$\sum_{i=1}^{1} i = \frac{(1)((1)+1)}{2} = 1,$$

which holds. Second, we assume the formula holds for $n = k$, so that we assume

$$1 + 2 + \cdots + k = \sum_{i=1}^{k} i = \frac{k((k)+1)}{2}.$$

We want to show that the formula also holds for $n = k + 1$. Consider

$$1 + 2 + \cdots + k + (k + 1) = (1 + 2 + \cdots + k) + (k + 1).$$

Solution (continued). We have

$$1 + 2 + \cdots + k + (k + 1)$$

$$= \left( \sum_{i=1}^{k} i \right) + (k + 1)$$

$$= \left( \frac{k(k+1)}{2} \right) + (k + 1)$$

by the induction hypothesis

$$= \frac{k(k + 1) + 2(k + 1)}{2} = \frac{k(k + 1) + 2(k + 1)}{2}$$

$$= \frac{(k + 1)((k + 1) + 1)}{2} = \frac{n(n + 1)}{2}$$

where $n = k + 1$.

So the formula holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed.

Example A.2.A

Example A.2.A. Prove that for differentiable functions of $x, u_1, u_2, \ldots, u_n$, we have

$$\frac{d}{dx}[u_1 + u_2 + \cdots + u_n] = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}.$$  

**Proof.** First, we check the formula for $n = 1$. This gives $\frac{d}{dx}[u_1] = \frac{du_1}{dx}$, which holds. Notice also that $\frac{d}{dx}[u_1 + u_2] = \frac{du_1}{dx} + \frac{du_2}{dx}$ by the Derivative Sum Rule (Theorem 3.3.E), so that the result also holds for $n = 2$.

Second, we assume the formula holds for $n = k$, so that we assume

$$\frac{d}{dx}[u_1 + u_2 + \cdots + u_k] = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx}.$$  

We want to show that the formula also holds for $n = k + 1$. Consider

$$\frac{d}{dx}[u_1 + u_2 + \cdots + u_k + u_{k+1}] = \frac{d}{dx}[(u_1 + u_2 + \cdots + u_k) + u_{k+1}].$$
Example A.2.A (continued)

Prove (continued). We have \( \frac{d}{dx}[u_1 + u_2 + \cdots + u_k + u_{k+1}] \)

\[
= \frac{d}{dx}[(u_1 + u_2 + \cdots + u_k) + u_{k+1}] \\
= \frac{d}{dx}[(u_1 + u_2 + \cdots + u_k)] + \frac{d}{dx}[u_{k+1}] \text{ since the result holds for } n = 2 \text{ functions} \\
= \left( \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx} \right) + \frac{d}{dx}[u_{k+1}] \\
\text{by the induction hypothesis} \\
= \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx} + \frac{du_{k+1}}{dx} \\
= \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx} + \frac{du_{k+1}}{dx} \text{ where } n = k + 1,
\]

so the result holds for \( n = k + 1 \) and, by the mathematical induction principle, the formula holds for all \( n \in \mathbb{N} \), as claimed.

Exercise A.2.2

Exercise A.2.2. Prove that if \( r \neq 1 \) then \( 1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} \) for every natural number \( n \in \mathbb{N} \).

Proof. First, we check the formula for \( n = 1 \). This gives
\[
1 + r = \frac{1-r^2}{1-r} = \frac{(1-r)(1+r)}{1-r} = 1 + r, \text{ which holds. Second, we assume} \\
1 + r + r^2 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}. \text{ We want to show that the formula also holds for } n = k + 1. \text{ Consider} \\
1 + r + r^2 + \cdots + r^k + r^{k+1} = (1 + r + r^2 + \cdots + r^k) + r^{k+1}.
\]

Exercise A.2.2 (continued)

Exercise A.2.2. Prove that if \( r \neq 1 \) then \( 1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} \) for every natural number \( n \in \mathbb{N} \).

Proof (continued). We have \( 1 + r + r^2 + \cdots + r^k + r^{k+1} \)

\[
= (1 + r + r^2 + \cdots + r^k) + r^{k+1} \\
= \left( \frac{1 - r^{k+1}}{1 - r} \right) + r^{k+1} \text{ by the induction hypothesis} \\
= \frac{1 - r^{k+1}}{1 - r} + \frac{r^{k+1}(1 - r)}{1 - r} = \frac{(1 - r^{k+1}) + (r^{k+1} - r^{k+2})}{1 - r} \\
\text{where } n = k + 1,
\]

so the result holds for \( n = k + 1 \) and, by the mathematical induction principle, the formula holds for all \( n \in \mathbb{N} \), as claimed.

Exercise A.2.9

Exercise A.2.9. Sums of Squares

Prove Theorem 5.2.B(2):
\[
\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6} \text{ for all } n \in \mathbb{N}.
\]

Proof. First, we check the formula for \( n = 1 \). This gives
\[
1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{6}{6} = 1, \text{ which holds. Second, we assume} \\
\text{the formula holds for } n = k, \text{ so that we assume} \\
1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}. \text{ We want to show that the formula also holds for } n = k + 1. \text{ Consider} \\
1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = (1^2 + 2^2 + \cdots + k^2) + (k + 1)^2.
\]
Exercise A.2.9 (continuous)

Proof (continued). We have $1^2 + 2^2 + \cdots + k^2 + (k + 1)^2$

\[
= \left(\frac{k(k + 1)(2k + 1)}{6}\right) + (k + 1)^2 \text{ by the induction hypothesis}
\]

\[
= \frac{k(k + 1)(2k + 1)}{6} + \frac{6(k + 1)^2}{6} = \frac{k(k + 1)(2k + 1) + 6(k + 1)^2}{6}
\]

\[
= \frac{(k + 1)(2k^2 + 7k + 6)}{6} = \frac{(k + 1)(2k + 1)(k + 2)(2k + 3)}{6}
\]

\[
= \frac{(k + 1)((k + 1) + 1)(2(k + 1) + 1)}{6} = \frac{n(n + 1)(2n + 1)}{6} \quad \text{where } n = k + 1,
\]

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. \qed

Exercise A.2.10

Exercise A.2.10. Sums of Cubes

Prove Theorem 5.2.B(3):

\[
\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n + 1)}{2}\right)^2 \text{ for all } n \in \mathbb{N}.
\]

Solution. First, we check the formula for $n = 1$. This gives

\[
1^3 = \left(\frac{1(1 + 1)}{2}\right)^2 = 1^2, \text{ which holds. Second, we assume the formula}
\]

holds for $n = k$, so that we assume $1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k + 1)}{2}\right)^2$.

We want to show that the formula also holds for $n = k + 1$. Consider

\[
1^3 + 2^3 + \cdots + k^3 + (k + 1)^3 = \left(\frac{k(k + 1)}{2}\right)^2 + (k + 1)^3
\]

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. \qed

Exercise A.2.11

Exercise A.2.11. Proof of Theorem 5.2.A


1 Sum Rule: $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

2 Difference Rule: $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

3 Constant Multiple Rule: $\sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$

4 Constant Value Rule: $\sum_{i=1}^{n} c = nc$
Exercise A.2.11 (continued 1)

**Sum Rule:** \( \sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \)

**Solution.** First, we check the formula for \( n = 1 \). This gives \((a_1 + b_1) = (a_1) + (b_1)\), which holds. Second, we assume the formula holds for \( n = k \), so that we assume \( \sum_{i=1}^{k} (a_i + b_i) = \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i \). We want to show that the formula also holds for \( n = k + 1 \). Consider \( \sum_{i=1}^{k+1} (a_i + b_i) = \left( \sum_{i=1}^{k} (a_i + b_i) \right) + (a_{k+1} + b_{k+1}) \).

Exercise A.2.11 (continued 2)

**Solution (continued).** We have \( \sum_{i=1}^{k+1} (a_i + b_i) \)
\[= \left( \sum_{i=1}^{k} (a_i + b_i) \right) + (a_{k+1} + b_{k+1}) \]
\[= \left( \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i \right) + (a_{k+1} + b_{k+1}) \] by the induction hypothesis
\[= \left( \sum_{i=1}^{k} a_i + a_{k+1} \right) + \left( \sum_{i=1}^{k} b_i + b_{k+1} \right) \]
by associativity and commutivity of addition in \( \mathbb{R} \)
\[= \sum_{i=1}^{k+1} a_i + \sum_{i=1}^{k+1} b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \]
where \( n = k + 1 \), so the result holds for \( n = k + 1 \) and, by the mathematical induction principle, the formula holds for all \( n \in \mathbb{N} \), as claimed. \( \square \)

Exercise A.2.11 (continued 3)

**Difference Rule:** \( \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \)

**Solution.** First, we check the formula for \( n = 1 \). This gives \((a_1 - b_1) = (a_1) - (b_1)\), which holds. Second, we assume the formula holds for \( n = k \), so that we assume \( \sum_{i=1}^{k} (a_i - b_i) = \sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i \). We want to show that the formula also holds for \( n = k + 1 \). Consider \( \sum_{i=1}^{k+1} (a_i - b_i) = \left( \sum_{i=1}^{k} (a_i - b_i) \right) + (a_{k+1} - b_{k+1}) \).

Exercise A.2.11 (continued 4)

**Solution (continued).** We have \( \sum_{i=1}^{k+1} (a_i - b_i) \)
\[= \left( \sum_{i=1}^{k} (a_i - b_i) \right) + (a_{k+1} - b_{k+1}) \]
\[= \left( \sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i \right) + (a_{k+1} - b_{k+1}) \] by the induction hypothesis
\[= \left( \sum_{i=1}^{k} a_i + a_{k+1} \right) - \left( \sum_{i=1}^{k} b_i + b_{k+1} \right) \]
by associativity and commutivity of addition in \( \mathbb{R} \)
\[= \sum_{i=1}^{k+1} a_i - \sum_{i=1}^{k+1} b_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \]
where \( n = k + 1 \), so the result holds for \( n = k + 1 \) and, by the mathematical induction principle, the formula holds for all \( n \in \mathbb{N} \), as claimed. \( \square \)
Exercise A.2.11 (continued 5)

Constant Multiple Rule: \[ \sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i \]

Solution. First, we check the formula for \( n = 1 \). This gives \((ca_1) = c(a_1)\), which holds. Second, we assume the formula holds for \( n = k \), so that we assume \( \sum_{i=1}^{k} c a_i = c \sum_{i=1}^{k} a_i \). We want to show that the formula also holds for \( n = k + 1 \). Consider \( \sum_{i=1}^{k+1} c a_i = \sum_{i=1}^{k} c a_i + c a_{k+1} \).

Exercise A.2.11 (continued 7)

Constant Value Rule: \[ \sum_{i=1}^{n} c = nc \]

Solution. First, we check the formula for \( n = 1 \). This gives \((c) = 1c\), which holds. Second, we assume the formula holds for \( n = k \), so that we assume \( \sum_{i=1}^{k} c = kc \). We want to show that the formula also holds for \( n = k + 1 \). Consider \( \sum_{i=1}^{k+1} c = \left( \sum_{i=1}^{k} c \right) + c \).

Exercise A.2.11 (continued 6)

Solution (continued). We have \( \sum_{i=1}^{k+1} c a_i \)

\[ = \sum_{i=1}^{k} c a_i + c a_{k+1} \]

\[ = c \sum_{i=1}^{k} a_i + c a_{k+1} \text{ by the induction hypothesis} \]

\[ = c \left( \sum_{i=1}^{k} a_i + a_{k+1} \right) \text{ since multiplication distributes over addition in } \mathbb{R} \]

\[ = c \left( \sum_{i=1}^{k+1} a_i \right) = c \sum_{i=1}^{n} a_i \text{ where } n = k + 1, \]

so the result holds for \( n = k + 1 \) and, by the mathematical induction principle, the formula holds for all \( n \in \mathbb{N} \), as claimed. \( \square \)

Exercise A.2.11 (continued 8)

Solution (continued). We have \( \sum_{i=1}^{k+1} c \)

\[ = \left( \sum_{i=1}^{k} c \right) + c \]

\[ = kc + c \text{ by the induction hypothesis} \]

\[ = (k+1)c \text{ since multiplication distributes over addition in } \mathbb{R} \]

\[ = nc \text{ where } n = k + 1, \]

so the result holds for \( n = k + 1 \) and, by the mathematical induction principle, the formula holds for all \( n \in \mathbb{N} \), as claimed. \( \square \)
Example A.2.B. General Product Rule

Example A.2.B. Prove the General Product Rule (see Exercise 3.3.77 for motivation of this result): For differentiable functions $u_1, u_2, \ldots, u_n$, we have that the derivative of the product $u_1 u_2 \cdots u_n$ exists and

$$
\frac{d}{dx}[(u_1)(u_2)\cdots(u_n)] = [u'_1](u_2)(u_3)\cdots(u_{n-1})(u_n) \\
+ (u_1)[u'_2](u_3)\cdots(u_{n-1})(u_n) \\
+ (u_1)(u_2)[u'_3]\cdots(u_{n-1})(u_n) + \cdots \\
+ (u_1)(u_2)(u_3)\cdots[u'_{n-1}](u_n) \\
+ (u_1)(u_2)(u_3)\cdots(u_{n-1})[u'_n].
$$

Proof. We introduce a product notation, similar to the summation notation: $u_1 u_2 \cdots u_n = \prod_{i=1}^n u_i$. We can then express the claim of this theorem as

$$
\frac{d}{dx} \left[ \prod_{i=1}^n u_i \right] = \sum_{j=1}^n u'_j \prod_{i=1, i \neq j}^n u_i.
$$

First, we check the formula for $n = 1$. This gives

$$
\frac{d}{dx}[u_1] = \sum_{j=1}^1 u'_j \prod_{i=1, i \neq j}^1 u_i = u'_1,
$$

which holds. For clarity, we also check the formula for $n = 2$. This gives

$$
\frac{d}{dx}[u_1 u_2] = \sum_{j=1}^2 u'_j \prod_{i=1, i \neq j}^2 u_i = [u'_1](u_2) + (u_1)[u'_2],
$$

which holds by the Derivative Product Rule (Theorem 3.3.G).

Example A.2.B (continued 2)

Proof (continued). Next, we assume the formula holds for $n = k$, so that we assume

$$
\frac{d}{dx} \left[ \prod_{i=1}^k u_i \right] = \sum_{j=1}^k u'_j \prod_{i=1, i \neq j}^k u_i.
$$

We want to show that the formula also holds for $n = k + 1$. Consider

$$
\frac{d}{dx} \left[ \prod_{i=1}^{k+1} u_i \right] = \frac{d}{dx} \left[ \left( \prod_{i=1}^k u_i \right) u_{k+1} \right]
$$

$$
= \frac{d}{dx} \left[ \prod_{i=1}^k u_i \right] (u_{k+1}) + \left( \prod_{i=1}^k u_i \right) [u'_{k+1}]
$$

by the Derivative Rule for Products (Theorem 3.3.G)

$$
= \sum_{j=1}^k u'_j \prod_{i=1, i \neq j}^k u_i (u_{k+1}) + \left( \prod_{i=1}^k u_i \right) [u'_{k+1}]
$$

by induction hypothesis

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed.

Example A.2.B (continued 3)