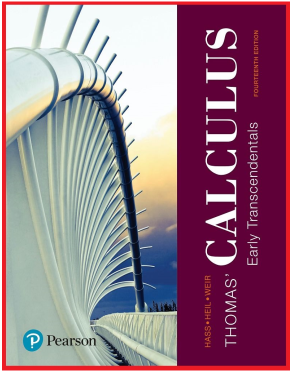
Calculus 1

Appendices

A.2. Mathematical Induction—Examples and Proofs



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Example A.2.1

Example A.2.1. Use mathematical induction to prove that for natural number $n \in \mathbb{N}$,

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Prove. First, we check the formula for $n = 1$. This gives $\sum_{i=1}^{1} i = 1 = \frac{(1)((1)+1)}{2} = 1$, which holds.

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Prove. First, we check the formula for $n = 1$. This gives $\sum_{i=1}^{1} i = 1 = \frac{(1)((1)+1)}{2} = 1$, which holds. Second, we assume the formula holds for $n = k$, so that we assume

$$1 + 2 + \cdots + k = \sum_{i=1}^{k} i = \frac{(k)((k)+1)}{2}.$$

We want to show that the formula also holds for $n = k + 1$. Consider $1 + 2 + \cdots + k + (k+1) = (1 + 2 + \cdots + k) + (k+1)$.

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We want to show that the formula also holds for $n = k + 1$. Consider $1 + 2 + \cdots + k + (k+1) = (1 + 2 + \cdots + k) + (k+1)$.

Example A.2.1 (solution)

Solution (continued). We have

$$
\begin{aligned}
& (1+2+\cdots+k)+(k+1) \\
= \quad & \left(\sum_{i=1}^{k} i\right)+(k+1) \\
= \quad & \left(\frac{k(k+1)}{2}\right)+(k+1) \text{ by the induction hypothesis} \\
= \quad & \frac{k(k+1)}{2}+\frac{2(k+1)}{2}=\frac{k(k+1)+2(k+1)}{2}=\frac{(k+1)(k+2)}{2} \\
= \quad & \frac{(k+1)((k+1)+1)}{2}=\frac{n(n+1)}{2} \text{ where } n=k+1.
\end{aligned}
$$

So the formula holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Example A.2.A

Example A.2.A. Prove that for differentiable functions of x, $u_1, u_2, \ldots, u_n$, we have

$$\frac{d}{dx}[u_1 + u_2 + \cdots + u_n] = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}.$$

Proof. First, we check the formula for $n = 1$. This gives $\frac{d}{dx}[u_1] = \frac{du_1}{dx}$, which holds. Notice also that $\frac{d}{dx}[u_1 + u_2] = \frac{du_1}{dx} + \frac{du_2}{dx}$ by the Derivative Sum Rule (Theorem 3.3.E), so that the result also holds for $n = 2$.

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Second, we assume the formula holds for $n = k$, so that we assume $\frac{d}{dx}[u_1 + u_2 + \cdots + u_k] = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx}$. We want to show that the formula also holds for $n = k + 1$. Consider $\frac{d}{dx}[u_1 + u_2 + \cdots + u_k + u_{k+1}] = \frac{d}{dx}[(u_1 + u_2 + \cdots + u_k) + u_{k+1}]$.

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Example A.2.A (continued)

Prove (continued). We have $\frac{d}{dx}[u_1 + u_2 + \cdots + u_k + u_{k+1}]$

$$
\begin{aligned}
&= \frac{d}{dx}[(u_1 + u_2 + \cdots + u_k) + u_{k+1}] \\
&= \frac{d}{dx}[(u_1 + u_2 + \cdots + u_k)] + \frac{d}{dx}[u_{k+1}] \text{ since the result} \\
&\quad \text{holds for } n = 2 \text{ functions} \\
&= \left(\frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx}\right) + \frac{d}{dx}[u_{k+1}] \\
&\quad \text{by the induction hypothesis} \\
&= \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx} + \frac{du_{k+1}}{dx} \\
&= \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx} \text{ where } n = k + 1,
\end{aligned}
$$

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Exercise A.2.2

Exercise A.2.2. Prove that if $r \neq 1$ then $1 + r + r^2 + \cdots + r^n = \dfrac{1 - r^{n+1}}{1 - r}$ for every natural number $n \in \mathbb{N}$.

Proof. First, we check the formula for $n = 1$. This gives $1 + r = \frac{1-r^2}{1-r} = \frac{(1-r)(1+r)}{1-r} = 1 + r$, which holds.

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Exercise A.2.2 (continued)

Exercise A.2.2. Prove that if $r \neq 1$ then $1 + r + r^2 + \cdots + r^n = \dfrac{1 - r^{n+1}}{1 - r}$ for every natural number $n \in \mathbb{N}$.

Proof (continued). We have $1 + r + r^2 + \cdots + r^k + r^{k+1}$

$$\begin{aligned} &= (1 + r + r^2 + \cdots + r^k) + r^{k+1} \\ &= \left(\frac{1 - r^{k+1}}{1 - r}\right) + r^{k+1} \text{ by the induction hypothesis} \\ &= \frac{1 - r^{k+1}}{1 - r} + \frac{r^{k+1}(1 - r)}{1 - r} = \frac{(1 - r^{k+1}) + (r^{k+1} - r^{k+2})}{1 - r} \\ &= \frac{1 - r^{k+2}}{1 - r} = \frac{1 - r^{n+1}}{1 - r} \text{ where } n = k + 1, \end{aligned}$$

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Exercise A.2.2 (continued)

Exercise A.2.2. Prove that if $r \neq 1$ then $1 + r + r^2 + \cdots + r^n = \dfrac{1 - r^{n+1}}{1 - r}$ for every natural number $n \in \mathbb{N}$.

Proof (continued). We have $1 + r + r^2 + \cdots + r^k + r^{k+1}$

$$\begin{aligned}
&= (1 + r + r^2 + \cdots + r^k) + r^{k+1} \\
&= \left(\frac{1 - r^{k+1}}{1 - r}\right) + r^{k+1} \text{ by the induction hypothesis} \\
&= \frac{1 - r^{k+1}}{1 - r} + \frac{r^{k+1}(1 - r)}{1 - r} = \frac{(1 - r^{k+1}) + (r^{k+1} - r^{k+2})}{1 - r} \\
&= \frac{1 - r^{k+2}}{1 - r} = \frac{1 - r^{n+1}}{1 - r} \text{ where } n = k + 1,
\end{aligned}$$

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Exercise A.2.9

Exercise A.2.9. Sums of Squares
Prove Theorem 5.2.B(2):

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \text{ for all } n \in \mathbb{N}.$$

Proof. First, we check the formula for $n = 1$. This gives $1^2 = \frac{(1)((1)+1)(2(1)+1)}{6} = \frac{6}{6} = 1$, which holds.

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Exercise A.2.9 (continuous)

Proof (continued). We have $1^2 + 2^2 + \cdots + k^2 + (k+1)^2$

$$\begin{aligned}
&= (1^2 + 2^2 + \cdots + k^2) + (k+1)^2 \\
&= \left(\frac{k(k+1)(2k+1)}{6}\right) + (k+1)^2 \text{ by the induction hypothesis} \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} = \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} = \frac{n(n+1)(2n+1)}{6} \text{ where } n = k+1,
\end{aligned}$$

so the result holds for $n = k+1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Exercise A.2.10

Exercise A.2.10. Sums of Cubes
Prove Theorem 5.2.B(3):

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \text{ for all } n \in \mathbb{N}.$$

Solution. First, we check the formula for $n = 1$. This gives $1^3 = \left(\frac{(1)((1)+1)}{2}\right)^2 = 1^2$, which holds.

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Solution. First, we check the formula for $n = 1$. This gives $1^3 = \left(\frac{(1)((1)+1)}{2}\right)^2 = 1^2$, which holds. Second, we assume the formula holds for $n = k$, so that we assume $1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$. We want to show that the formula also holds for $n = k + 1$. Consider $1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = (1^3 + 2^3 + \cdots + k^3) + (k+1)^3$.

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Prove Theorem 5.2.B(3):

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \text{ for all } n \in \mathbb{N}.$$

Solution. First, we check the formula for $n = 1$. This gives $1^3 = \left(\frac{(1)((1)+1)}{2}\right)^2 = 1^2$, which holds. Second, we assume the formula holds for $n = k$, so that we assume $1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$.
We want to show that the formula also holds for $n = k + 1$. Consider $1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = (1^3 + 2^3 + \cdots + k^3) + (k+1)^3$.

Exercise A.2.10 (continued)

Solution. We have $1^3 + 2^3 + \cdots + k^3 + (k+1)^3$

$$\begin{aligned}
&= (1^3 + 2^3 + \cdots + k^3) + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \text{ by the induction hypothesis} \\
&= \left(\frac{k(k+1)}{2}\right)^2 + \frac{4(k+1)^3}{4} = \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} = \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} = \frac{(k+1)^2((k+1)+1)^2}{4} \\
&= \left(\frac{(k+1)((k+1)+1)}{2}\right)^2 = \left(\frac{n(n+1)}{2}\right)^2 \text{ where } n = k+1,
\end{aligned}$$

so the result holds for $n = k+1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Exercise A.2.11

Exercise A.2.11. Prove Theorem 5.2.A, "Algebra for Finite Sums."

1. *Sum Rule*: $\displaystyle\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
2. *Difference Rule*: $\displaystyle\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$
3. *Constant Multiple Rule*: $\displaystyle\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
4. *Constant Value Rule*: $\displaystyle\sum_{i=1}^{n} c = nc$

Exercise A.2.11 (continued 1)

$$\textit{Sum Rule: } \sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

Solution. First, we check the formula for $n = 1$. This gives $(a_1 + b_1) = (a_1) + (b_1)$, which holds. Second, we assume the formula holds for $n = k$, so that we assume $\sum_{i=1}^{k}(a_i + b_i) = \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i$. We want to show that the formula also holds for $n = k + 1$. Consider $\sum_{i=1}^{k+1}(a_i + b_i) = \left(\sum_{i=1}^{k}(a_i + b_i)\right) + (a_{k+1} + b_{k+1})$.

Exercise A.2.11 (continued 1)

$$\textit{Sum Rule}: \sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

Solution. First, we check the formula for $n = 1$. This gives $(a_1 + b_1) = (a_1) + (b_1)$, which holds. Second, we assume the formula holds for $n = k$, so that we assume $\displaystyle\sum_{i=1}^{k}(a_i + b_i) = \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i$. We want to show that the formula also holds for $n = k + 1$. Consider $\displaystyle\sum_{i=1}^{k+1}(a_i + b_i) = \left(\sum_{i=1}^{k}(a_i + b_i)\right) + (a_{k+1} + b_{k+1})$.

Exercise A.2.11 (continued 2)

Solution (continued). We have $\displaystyle\sum_{i=1}^{k+1}(a_i + b_i)$

$$= \left(\sum_{i=1}^{k}(a_i + b_i)\right) + (a_{k+1} + b_{k+1})$$

$$= \left(\sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i\right) + (a_{k+1} + b_{k+1}) \text{ by the induction hypothesis}$$

$$= \left(\sum_{i=1}^{k} a_i + a_{k+1}\right) + \left(\sum_{i=1}^{k} b_i + b_{k+1}\right)$$

by associativity and commutivity of addition in $\mathbb{R}$

$$= \sum_{i=1}^{k+1} a_i + \sum_{i=1}^{k+1} b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \text{ where } n = k+1,$$

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Exercise A.2.11 (continued 3)

$$\textit{Difference Rule}: \sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

Solution. First, we check the formula for $n = 1$. This gives $(a_1 - b_1) = (a_1) - (b_1)$, which holds. Second, we assume the formula holds for $n = k$, so that we assume $\sum_{i=1}^{k}(a_i - b_i) = \sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i$. We want to show that the formula also holds for $n = k + 1$. Consider $\sum_{i=1}^{k+1}(a_i - b_i) = \left(\sum_{i=1}^{k}(a_i - b_i)\right) + (a_{k+1} - b_{k+1})$.

Exercise A.2.11 (continued 3)

$$\textit{Difference Rule}: \sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

Solution. First, we check the formula for $n = 1$. This gives $(a_1 - b_1) = (a_1) - (b_1)$, which holds. Second, we assume the formula holds for $n = k$, so that we assume $\sum_{i=1}^{k}(a_i - b_i) = \sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i$. We want to show that the formula also holds for $n = k + 1$. Consider $\sum_{i=1}^{k+1}(a_i - b_i) = \left(\sum_{i=1}^{k}(a_i - b_i)\right) + (a_{k+1} - b_{k+1})$.

Exercise A.2.11 (continued 4)

Solution (continued). We have $\sum_{i=1}^{k+1}(a_i - b_i)$

$$= \left(\sum_{i=1}^{k}(a_i - b_i)\right) + (a_{k+1} - b_{k+1})$$

$$= \left(\sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i\right) + (a_{k+1} - b_{k+1}) \text{ by the induction hypothesis}$$

$$= \left(\sum_{i=1}^{k} a_i + a_{k+1}\right) - \left(\sum_{i=1}^{k} b_i + b_{k+1}\right)$$

by associativity and commutivity of addition in $\mathbb{R}$

$$= \sum_{i=1}^{k+1} a_i - \sum_{i=1}^{k+1} b_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \text{ where } n = k+1,$$

so the result holds for $n = k+1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Exercise A.2.11 (continued 5)

$$\textit{Constant Multiple Rule}: \sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

Solution. First, we check the formula for $n = 1$. This gives $(ca_1) = c(a_1)$, which holds. Second, we assume the formula holds for $n = k$, so that we assume $\sum_{i=1}^{k} ca_i = c\sum_{i=1}^{k} a_i$. We want to show that the formula also holds for $n = k + 1$. Consider $\sum_{i=1}^{k+1} ca_i = \sum_{i=1}^{k} ca_i + ca_{k+1}$.

Exercise A.2.11 (continued 5)

$$\textit{Constant Multiple Rule}: \ \sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$$

Solution. First, we check the formula for $n = 1$. This gives $(ca_1) = c(a_1)$, which holds. Second, we assume the formula holds for $n = k$, so that we assume $\displaystyle\sum_{i=1}^{k} ca_i = c \sum_{i=1}^{k} a_i$. We want to show that the formula also holds for $n = k + 1$. Consider $\displaystyle\sum_{i=1}^{k+1} ca_i = \sum_{i=1}^{k} ca_i + ca_{k+1}$.

Exercise A.2.11 (continued 6)

Solution (continued). We have $\displaystyle\sum_{i=1}^{k+1} ca_i$

$$
\begin{aligned}
&= \sum_{i=1}^{k} ca_i + ca_{k+1} \\
&= c\sum_{i=1}^{k} a_i + ca_{k+1} \text{ by the induction hypothesis} \\
&= c\left(\sum_{i=1}^{k} a_i + a_{k+1}\right) \text{ since multiplication distributes over addition in } \mathbb{R} \\
&= c\left(\sum_{i=1}^{k+1} a_i\right) = c\sum_{i=1}^{n} a_i \text{ where } n = k+1,
\end{aligned}
$$

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Exercise A.2.11 (continued 7)

$$\textit{Constant Value Rule}: \sum_{i=1}^{n} c = nc$$

Solution. First, we check the formula for $n = 1$. This gives $(c) = 1c$, which holds. Second, we assume the formula holds for $n = k$, so that we assume $\displaystyle\sum_{i=1}^{k} c = kc$. We want to show that the formula also holds for $n = k + 1$. Consider $\displaystyle\sum_{i=1}^{k+1} c = \left(\sum_{i=1}^{k} c\right) + c$.

Exercise A.2.11 (continued 7)

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Exercise A.2.11 (continued 8)

Solution (continued). We have $\displaystyle\sum_{i=1}^{k+1} c$

$$
\begin{aligned}
&= \left(\sum_{i=1}^{k} c\right) + c \\
&= kc + c \text{ by the induction hypothesis} \\
&= (k+1)c \text{ since multiplication distributes over addition in } \mathbb{R} \\
&= nc \text{ where } n = k+1,
\end{aligned}
$$

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □

Example A.2.B. General Product Rule

Example A.2.B. Prove the General Product Rule (see Exercise 3.3.77 for motivation of this result): For differentiable functions $u_1, u_2, \ldots, u_n$, we have that the derivative of the product $u_1 u_2 \cdots u_n$ exists and

$$\begin{aligned} \frac{d}{dx}[(u_1)(u_2)\cdots(u_n)] &= [u_1'](u_2)(u_3)\cdots(u_{n-1})(u_n) \\ &\quad +(u_1)[u_2'](u_3)\cdots(u_{n-1})(u_n) \\ &\quad +(u_1)(u_2)[u_3']\cdots(u_{n-1})(u_n)+\cdots \\ &\quad +(u_1)(u_2)(u_3)\cdots[u_{n-1}'](u_n) \\ &\quad +(u_1)(u_2)(u_3)\cdots(u_{n-1})[u_n']. \end{aligned}$$

Example A.2.B (continued 1)

Proof. We introduce a product notation, similar to the summation notation: $u_1 u_2 \cdots u_n = \prod_{i=1}^{n} u_i$. We can then express the claim of this theorem as

$$\frac{d}{dx}\left[\prod_{i=1}^{n} u_i\right] = \sum_{j=1}^{n} u_j' \prod_{i=1, i\neq j}^{n} u_i.$$

First, we check the formula for $n = 1$. This gives $\frac{d}{dx}[u_1] = \sum_{j=1}^{1} u_j' \prod_{i=1, i\neq 1}^{1} u_i = u_1'$, which holds. For clarity, we also check the formula for $n = 2$. This gives $\frac{d}{dx}[u_1 u_2] = \sum_{j=1}^{2} u_j' \prod_{i=1, i\neq j}^{2} u_i = [u_1'](u_2) + (u_1)[u_2']$, which holds by the Derivative Product Rule (Theorem 3.3.G).

Example A.2.B (continued 1)

Proof. We introduce a product notation, similar to the summation notation: $u_1 u_2 \cdots u_n = \prod_{i=1}^{n} u_i$. We can then express the claim of this theorem as

$$\frac{d}{dx}\left[\prod_{i=1}^{n} u_i\right] = \sum_{j=1}^{n} u_j' \prod_{i=1, i\neq j}^{n} u_i.$$

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Example A.2.B (continued 2)

Proof (continued). Next, we assume the formula holds for $n = k$, so that we assume $\frac{d}{dx}\left[\prod_{i=1}^{k} u_i\right] = \sum_{j=1}^{k} u_j' \prod_{i=1, i\neq j}^{k} u_i$. We want to show that the formula also holds for $n = k+1$. Consider

$$\frac{d}{dx}\left[\prod_{i=1}^{k+1} u_i\right] = \frac{d}{dx}\left[\left(\prod_{i=1}^{k} u_i\right) u_{k+1}\right]$$

$$= \frac{d}{dx}\left[\prod_{i=1}^{k} u_i\right](u_{k+1}) + \left(\prod_{i=1}^{k} u_i\right)[u_{k+1}'] \text{ by the Derivative Rule for Products (Theorem 3.3.G)}$$

$$= \left[\sum_{j=1}^{k} u_j' \prod_{i=1, i\neq j}^{k} u_i\right](u_{k+1}) + \left(\prod_{i=1}^{k} u_i\right)[u_{k+1}'] \text{ by induction hypothesis}$$

Example A.2.B (continued 2)

Proof (continued). Next, we assume the formula holds for $n = k$, so that we assume $\dfrac{d}{dx}\left[\prod_{i=1}^{k} u_i\right] = \sum_{j=1}^{k} u_j' \prod_{i=1, i\neq j}^{k} u_i$. We want to show that the formula also holds for $n = k+1$. Consider

$$\frac{d}{dx}\left[\prod_{i=1}^{k+1} u_i\right] = \frac{d}{dx}\left[\left(\prod_{i=1}^{k} u_i\right) u_{k+1}\right]$$

$$= \frac{d}{dx}\left[\prod_{i=1}^{k} u_i\right](u_{k+1}) + \left(\prod_{i=1}^{k} u_i\right)[u_{k+1}'] \text{ by the Derivative Rule}$$

for Products (Theorem 3.3.G)

$$= \left[\sum_{j=1}^{k} u_j' \prod_{i=1, i\neq j}^{k} u_i\right](u_{k+1}) + \left(\prod_{i=1}^{k} u_i\right)[u_{k+1}'] \text{ by induction hypothesis}$$

Example A.2.B (continued 3)

Proof (continued). $\ldots \dfrac{d}{dx}\left[\prod_{i=1}^{k+1} u_i\right]$

$$\begin{aligned}
&= \left[\sum_{j=1}^{k} u_j' \prod_{i=1, i\neq j}^{k} u_i\right](u_{k+1}) + \left(\prod_{i=1}^{k} u_i\right)[u_{k+1}'] \\
&= \sum_{j=1}^{k} u_j' \prod_{i=1, i\neq j}^{k+1} u_i + u_{k+1}' \prod_{i=1, i\neq k+1}^{k+1} u_i \\
&= \sum_{j=1}^{k+1} u_j \prod_{i=1, i\neq j}^{k+1} u_i = \sum_{j=1}^{n} u_j \prod_{i=1, i\neq j}^{n} u_i \text{ where } n = k+1,
\end{aligned}$$

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. □