Appendices
A.2. Mathematical Induction—Examples and Proofs
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Example A.2.1

Example A.2.1. Use mathematical induction to prove that for natural number \( n \in \mathbb{N} \),

\[
1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.
\]

Prove. First, we check the formula for \( n = 1 \). This gives

\[
\sum_{i=1}^{1} i = 1 = \frac{(1)((1) + 1)}{2} = 1,
\]

which holds.
Example A.2.1. Use mathematical induction to prove that for natural number $n \in \mathbb{N}$,

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$  

Prove. First, we check the formula for $n = 1$. This gives

$$\sum_{i=1}^{1} i = 1 = \frac{(1)((1) + 1)}{2} = 1,$$

which holds. Second, we assume the formula holds for $n = k$, so that we assume

$$1 + 2 + \cdots + k = \sum_{i=1}^{k} i = \frac{(k)((k) + 1)}{2}.$$  

We want to show that the formula also holds for $n = k + 1$. Consider

$$1 + 2 + \cdots + k + (k + 1) = (1 + 2 + \cdots + k) + (k + 1).$$
Example A.2.1

Example A.2.1. Use mathematical induction to prove that for natural number \( n \in \mathbb{N} \),

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Prove. First, we check the formula for \( n = 1 \). This gives

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which holds. Second, we assume the formula holds for \( n = k \), so that we assume

\[
1 + 2 + \cdots + k = \sum_{i=1}^{k} i = \frac{(k)((k) + 1)}{2}.
\]

We want to show that the formula also holds for \( n = k + 1 \). Consider

\[
1 + 2 + \cdots + k + (k + 1) = (1 + 2 + \cdots + k) + (k + 1).
\]
Solution (continued). We have

\[(1 + 2 + \cdots + k) + (k + 1)\]

\[= \left( \sum_{i=1}^{k} i \right) + (k + 1)\]

\[= \left( \frac{k(k + 1)}{2} \right) + (k + 1) \text{ by the induction hypothesis}\]

\[= \frac{k(k + 1)}{2} + \frac{2(k + 1)}{2} = \frac{k(k + 1) + 2(k + 1)}{2} = \frac{(k + 1)(k + 2)}{2}\]

\[= \frac{(k + 1)((k + 1) + 1)}{2} = \frac{n(n + 1)}{2} \text{ where } n = k + 1.\]

So the formula holds for \(n = k + 1\) and, by the mathematical induction principle, the formula holds for all \(n \in \mathbb{N}\), as claimed.
Example A.2.A. Prove that for differentiable functions of $x$, $u_1, u_2, \ldots, u_n$, we have

$$
\frac{d}{dx}[u_1 + u_2 + \cdots + u_n] = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}.
$$

Proof. First, we check the formula for $n = 1$. This gives $\frac{d}{dx}[u_1] = \frac{du_1}{dx}$, which holds. Notice also that $\frac{d}{dx}[u_1 + u_2] = \frac{du_1}{dx} + \frac{du_2}{dx}$ by the Derivative Sum Rule (Theorem 3.3.E), so that the result also holds for $n = 2$. 
Example A.2.A. Prove that for differentiable functions of $x$, $u_1, u_2, \ldots, u_n$, we have

$$\frac{d}{dx}[u_1 + u_2 + \cdots + u_n] = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}.$$ 

Proof. First, we check the formula for $n = 1$. This gives $\frac{d}{dx}[u_1] = \frac{du_1}{dx}$, which holds. Notice also that $\frac{d}{dx}[u_1 + u_2] = \frac{du_1}{dx} + \frac{du_2}{dx}$ by the Derivative Sum Rule (Theorem 3.3.E), so that the result also holds for $n = 2$.

Second, we assume the formula holds for $n = k$, so that we assume

$$\frac{d}{dx}[u_1 + u_2 + \cdots + u_k] = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx}.$$ 

We want to show that the formula also holds for $n = k + 1$. Consider

$$\frac{d}{dx}[u_1 + u_2 + \cdots + u_k + u_{k+1}] = \frac{d}{dx}[(u_1 + u_2 + \cdots + u_k) + u_{k+1}].$$
Example A.2.A. Prove that for differentiable functions of $x$, $u_1, u_2, \ldots, u_n$, we have
\[
\frac{d}{dx} [u_1 + u_2 + \cdots + u_n] = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}.
\]

Proof. First, we check the formula for $n = 1$. This gives $\frac{d}{dx} [u_1] = \frac{du_1}{dx}$, which holds. Notice also that $\frac{d}{dx} [u_1 + u_2] = \frac{du_1}{dx} + \frac{du_2}{dx}$ by the Derivative Sum Rule (Theorem 3.3.E), so that the result also holds for $n = 2$.
Second, we assume the formula holds for $n = k$, so that we assume
\[
\frac{d}{dx} [u_1 + u_2 + \cdots + u_k] = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx}.
\]
We want to show that the formula also holds for $n = k + 1$. Consider
\[
\frac{d}{dx} [u_1 + u_2 + \cdots + u_k + u_{k+1}] = \frac{d}{dx} [(u_1 + u_2 + \cdots + u_k) + u_{k+1}].
\]
Example A.2.A (continued)

Prove (continued). We have \( \frac{d}{dx}[u_1 + u_2 + \cdots + u_k + u_{k+1}] \)

\[
= \frac{d}{dx}[(u_1 + u_2 + \cdots + u_k) + u_{k+1}] \\
= \frac{d}{dx}[(u_1 + u_2 + \cdots + u_k)] + \frac{d}{dx}[u_{k+1}] \text{ since the result holds for } n = 2 \text{ functions} \\
= \left( \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx} \right) + \frac{d}{dx}[u_{k+1}] \\
\text{by the induction hypothesis} \\
= \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx} + \frac{du_{k+1}}{dx} \\
= \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx} \text{ where } n = k + 1, \\
\text{so the result holds for } n = k + 1 \text{ and, by the mathematical induction principle, the formula holds for all } n \in \mathbb{N}, \text{ as claimed.} \]
Exercise A.2.2

Exercise A.2.2. Prove that if $r \neq 1$ then $1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$ for every natural number $n \in \mathbb{N}$.

Proof. First, we check the formula for $n = 1$. This gives $1 + r = \frac{1-r^2}{1-r} = \frac{(1-r)(1+r)}{1-r} = 1 + r$, which holds.
Exercise A.2.2.  Prove that if \( r \neq 1 \) then \( 1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} \) for every natural number \( n \in \mathbb{N} \).

**Proof.** First, we check the formula for \( n = 1 \). This gives
\[
1 + r = \frac{1-r^2}{1-r} = \frac{(1-r)(1+r)}{1-r} = 1 + r, \text{ which holds.}
\]
Second, we assume the formula holds for \( n = k \), so that we assume
\[
1 + r + r^2 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}. \]
We want to show that the formula also holds for \( n = k + 1 \). Consider
\[
1 + r + r^2 + \cdots + r^k + r^{k+1} = (1 + r + r^2 + \cdots + r^k) + r^{k+1}.
\]
Exercise A.2.2. Prove that if $r \neq 1$ then $1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$ for every natural number $n \in \mathbb{N}$.

Proof. First, we check the formula for $n = 1$. This gives

$$1 + r = \frac{1 - r^2}{1 - r} = \frac{(1-r)(1+r)}{1-r} = 1 + r,$$

which holds. Second, we assume the formula holds for $n = k$, so that we assume

$$1 + r + r^2 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}.$$

We want to show that the formula also holds for $n = k + 1$. Consider

$$1 + r + r^2 + \cdots + r^k + r^{k+1} = (1 + r + r^2 + \cdots + r^k) + r^{k+1}.$$
Exercise A.2.2. Prove that if $r \neq 1$ then $1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$ for every natural number $n \in \mathbb{N}$.

Proof (continued). We have $1 + r + r^2 + \cdots + r^k + r^{k+1}$

\[
= \left(1 + r + r^2 + \cdots + r^k\right) + r^{k+1} \\
= \left(\frac{1 - r^{k+1}}{1 - r}\right) + r^{k+1} \text{ by the induction hypothesis} \\
= \frac{1 - r^{k+1}}{1 - r} + \frac{r^{k+1}(1 - r)}{1 - r} = \frac{(1 - r^{k+1}) + (r^{k+1} - r^{k+2})}{1 - r} \\
= \frac{1 - r^{k+2}}{1 - r} = \frac{1 - r^{n+1}}{1 - r} \text{ where } n = k + 1,
\]

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. \qed
Exercise A.2.2. Prove that if $r \neq 1$ then $1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$ for every natural number $n \in \mathbb{N}$.

Proof (continued). We have $1 + r + r^2 + \cdots + r^k + r^{k+1}$

\[
= (1 + r + r^2 + \cdots + r^k) + r^{k+1}
\]

\[
= \left( \frac{1 - r^{k+1}}{1 - r} \right) + r^{k+1} \quad \text{by the induction hypothesis}
\]

\[
= \frac{1 - r^{k+1}}{1 - r} + \frac{r^{k+1}(1 - r)}{1 - r} = \frac{(1 - r^{k+1}) + (r^{k+1} - r^{k+2})}{1 - r}
\]

\[
= \frac{1 - r^{k+2}}{1 - r} = \frac{1 - r^{n+1}}{1 - r} \quad \text{where } n = k + 1,
\]

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed. \qed
Exercise A.2.9.

Exercise A.2.9. Sums of Squares

Prove Theorem 5.2.B(2):

\[ \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6} \]

for all \( n \in \mathbb{N} \).

Proof. First, we check the formula for \( n = 1 \). This gives

\[ 1^2 = \frac{(1)((1) + 1)(2(1) + 1)}{6} = \frac{6}{6} = 1, \]

which holds.
Exercise A.2.9. Sums of Squares

Prove Theorem 5.2.B(2):

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

for all $n \in \mathbb{N}$.

Proof. First, we check the formula for $n = 1$. This gives

$$1^2 = \frac{(1)((1) + 1)(2(1) + 1)}{6} = \frac{6}{6} = 1,$$

which holds. Second, we assume the formula holds for $n = k$, so that we assume

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k + 1)(2k + 1)}{6}.$$  

We want to show that the formula also holds for $n = k + 1$. Consider

$$1^2 + 2^2 + \cdots + k^2 + (k + 1)^2 = (1^2 + 2^2 + \cdots + k^2) + (k + 1)^2.$$
Exercise A.2.9. Sums of Squares

Prove Theorem 5.2.B(2):

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6} \quad \text{for all } n \in \mathbb{N}.$$ 

Proof. First, we check the formula for $n = 1$. This gives

$$1^2 = \frac{(1)((1) + 1)(2(1) + 1)}{6} = \frac{6}{6} = 1,$$

which holds. Second, we assume the formula holds for $n = k$, so that we assume

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k + 1)(2k + 1)}{6}.$$ 

We want to show that the formula also holds for $n = k + 1$. Consider

$$1^2 + 2^2 + \cdots + k^2 + (k + 1)^2 = (1^2 + 2^2 + \cdots + k^2) + (k + 1)^2.$$
Exercise A.2.9 (continuous)

Proof (continued). We have $1^2 + 2^2 + \cdots + k^2 + (k + 1)^2$

\[
= (1^2 + 2^2 + \cdots + k^2) + (k + 1)^2
\]

\[
= \left( \frac{k(k + 1)(2k + 1)}{6} \right) + (k + 1)^2 \text{ by the induction hypothesis}
\]

\[
= \frac{k(k + 1)(2k + 1)}{6} + \frac{6(k + 1)^2}{6} = \frac{k(k + 1)(2k + 1) + 6(k + 1)^2}{6}
\]

\[
= \frac{(k + 1)(k(2k + 1) + 6(k + 1))}{6} = \frac{(k + 1)(2k^2 + k + 6k + 6)}{6}
\]

\[
= \frac{(k + 1)(2k^2 + 7k + 6)}{6} = \frac{(k + 1)(k + 2)(2k + 3)}{6}
\]

\[
= \frac{(k + 1)((k + 1) + 1)(2(k + 1) + 1)}{6} = \frac{n(n + 1)(2n + 1)}{6} \quad \text{where } n = k + 1,
\]

so the result holds for $n = k + 1$ and, by the mathematical induction principle, the formula holds for all $n \in \mathbb{N}$, as claimed.
Exercise A.2.10

Exercise A.2.10. Sums of Cubes
Prove Theorem 5.2.B(3):

\[
\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \left( \frac{n(n+1)}{2} \right)^2 \quad \text{for all } n \in \mathbb{N}.
\]

Solution. First, we check the formula for \( n = 1 \). This gives

\[
1^3 = \left( \frac{(1)((1) + 1)}{2} \right)^2 = 1^2, \quad \text{which holds.}
\]
Exercise A.2.10. Sums of Cubes

Prove Theorem 5.2.B(3):

\[ \sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \left( \frac{n(n+1)}{2} \right)^2 \quad \text{for all } n \in \mathbb{N}. \]

Solution. First, we check the formula for \( n = 1 \). This gives

\[ 1^3 = \left( \frac{(1)(((1) + 1))}{2} \right)^2 = 1^2, \quad \text{which holds.} \]

Second, we assume the formula holds for \( n = k \), so that we assume \( 1^3 + 2^3 + \cdots + k^3 = \left( \frac{k(k+1)}{2} \right)^2 \).

We want to show that the formula also holds for \( n = k + 1 \). Consider

\[ 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = (1^3 + 2^3 + \cdots + k^3) + (k+1)^3. \]
Exercise A.2.10. Sums of Cubes

Prove Theorem 5.2.B(3):

\[
\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \left( \frac{n(n+1)}{2} \right)^2 \text{ for all } n \in \mathbb{N}.
\]

Solution. First, we check the formula for \( n = 1 \). This gives

\[1^3 = \left( \frac{1(1+1)}{2} \right)^2 = 1^2,\] which holds. Second, we assume the formula holds for \( n = k \), so that we assume

\[1^3 + 2^3 + \cdots + k^3 = \left( \frac{k(k+1)}{2} \right)^2.\]

We want to show that the formula also holds for \( n = k + 1 \). Consider

\[1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = (1^3 + 2^3 + \cdots + k^3) + (k+1)^3.\]
Exercise A.2.10. Sums of Cubes

**Solution.** We have 
\[ 1^3 + 2^3 + \cdots + k^3 + (k + 1)^3 \]

\[
= (1^3 + 2^3 + \cdots + k^3) + (k + 1)^3 \\
= \left( \frac{k(k + 1)}{2} \right)^2 + (k + 1)^3 \text{ by the induction hypothesis} \\
= \left( \frac{k(k + 1)}{2} \right)^2 + \frac{4(k + 1)^3}{4} = \frac{k^2(k + 1)^2 + 4(k + 1)^3}{4} \\
= \frac{(k + 1)^2(k^2 + 4(k + 1))}{4} = \frac{(k + 1)^2(k^2 + 4k + 4)}{4} \\
= \frac{(k + 1)^2(k + 2)^2}{4} = \frac{(k + 1)^2((k + 1) + 1)^2}{4} \\
= \left( \frac{(k + 1)((k + 1) + 1)}{2} \right)^2 = \left( \frac{n(n + 1)}{2} \right)^2 \text{ where } n = k + 1, \\
\]

so the result holds for \( n = k + 1 \) and, by the mathematical induction principle, the formula holds for all \( n \in \mathbb{N} \), as claimed.

1. **Sum Rule:** \[ \sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \]

2. **Difference Rule:** \[ \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \]

3. **Constant Multiple Rule:** \[ \sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i \]

4. **Constant Value Rule:** \[ \sum_{i=1}^{n} c = nc \]
Solution. First, we check the formula for $n = 1$. This gives
$(a_1 + b_1) = (a_1) + (b_1)$, which holds. Second, we assume the formula holds for $n = k$, so that we assume
$k\sum_{i=1}^{k} (a_i + b_i) = \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i$. We want to show that the formula also holds for $n = k + 1$. Consider $k+1\sum_{i=1}^{k+1} (a_i + b_i) = \left(\sum_{i=1}^{k} (a_i + b_i)\right) + (a_{k+1} + b_{k+1}).$
Exercise A.2.11 (continued 1)

Sum Rule: \[ \sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n}a_i + \sum_{i=1}^{n}b_i \]

Solution. First, we check the formula for \( n = 1 \). This gives \((a_1 + b_1) = (a_1) + (b_1)\), which holds. Second, we assume the formula holds for \( n = k \), so that we assume \( \sum_{i=1}^{k}(a_i + b_i) = \sum_{i=1}^{k}a_i + \sum_{i=1}^{k}b_i \). We want to show that the formula also holds for \( n = k + 1 \). Consider

\[ \sum_{i=1}^{k+1}(a_i + b_i) = \left( \sum_{i=1}^{k}(a_i + b_i) \right) + (a_{k+1} + b_{k+1}) \].
Solution (continued). We have \( \sum_{i=1}^{k+1} (a_i + b_i) \)

\[
\begin{align*}
&= \left( \sum_{i=1}^{k} (a_i + b_i) \right) + (a_{k+1} + b_{k+1}) \\
&= \left( \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i \right) + (a_{k+1} + b_{k+1}) \text{ by the induction hypothesis} \\
&= \left( \sum_{i=1}^{k} a_i + a_{k+1} \right) + \left( \sum_{i=1}^{k} b_i + b_{k+1} \right) \\
&\quad \text{by associativity and commutativity of addition in } \mathbb{R} \\
&= \sum_{i=1}^{k+1} a_i + \sum_{i=1}^{k+1} b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \text{ where } n = k + 1,
\end{align*}
\]

so the result holds for \( n = k + 1 \) and, by the mathematical induction principle, the formula holds for all \( n \in \mathbb{N} \), as claimed.
Exercise A.2.11 (continued 3)

Difference Rule: \[ \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \]

Solution. First, we check the formula for \( n = 1 \). This gives
\[ (a_1 - b_1) = (a_1) - (b_1), \]
which holds. Second, we assume the formula holds for \( n = k \), so that we assume \[ \sum_{i=1}^{k} (a_i - b_i) = \sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i. \]
We want to show that the formula also holds for \( n = k + 1 \). Consider
\[ \sum_{i=1}^{k+1} (a_i - b_i) = \left( \sum_{i=1}^{k} (a_i - b_i) \right) + (a_{k+1} - b_{k+1}). \]
Exercise A.2.11 (continued 3)

Difference Rule: \[ \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \]

Solution. First, we check the formula for \( n = 1 \). This gives \((a_1 - b_1) = (a_1) - (b_1)\), which holds. Second, we assume the formula holds for \( n = k \), so that we assume \( \sum_{i=1}^{k} (a_i - b_i) = \sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i \). We want to show that the formula also holds for \( n = k + 1 \). Consider \( \sum_{i=1}^{k+1} (a_i - b_i) = \left( \sum_{i=1}^{k} (a_i - b_i) \right) + (a_{k+1} - b_{k+1}) \).
Solution (continued). We have \( \sum_{i=1}^{k+1} (a_i - b_i) \)

\[
= \left( \sum_{i=1}^{k} (a_i - b_i) \right) + (a_{k+1} - b_{k+1})
\]

\[
= \left( \sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i \right) + (a_{k+1} - b_{k+1}) \text{ by the induction hypothesis}
\]

\[
= \left( \sum_{i=1}^{k} a_i + a_{k+1} \right) - \left( \sum_{i=1}^{k} b_i + b_{k+1} \right)
\]

by associativity and commutivity of addition in \( \mathbb{R} \)

\[
= \sum_{i=1}^{k+1} a_i - \sum_{i=1}^{k+1} b_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \text{ where } n = k + 1,
\]

so the result holds for \( n = k + 1 \) and, by the mathematical induction principle, the formula holds for all \( n \in \mathbb{N} \), as claimed.
Exercise A.2.11 (continued 5)

Constant Multiple Rule: \[ \sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i \]

Solution. First, we check the formula for \( n = 1 \). This gives \((ca_1) = c(a_1)\), which holds. Second, we assume the formula holds for \( n = k \), so that we assume \[ \sum_{i=1}^{k} ca_i = c \sum_{i=1}^{k} a_i. \] We want to show that the formula also holds for \( n = k + 1 \). Consider \[ \sum_{i=1}^{k+1} ca_i = \sum_{i=1}^{k} ca_i + ca_{k+1}. \]
Constant Multiple Rule: \[ \sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i \]

**Solution.** First, we check the formula for \( n = 1 \). This gives \((ca_1) = c(a_1)\), which holds. Second, we assume the formula holds for \( n = k \), so that we assume \( \sum_{i=1}^{k} c a_i = c \sum_{i=1}^{k} a_i \). We want to show that the formula also holds for \( n = k + 1 \). Consider \( \sum_{i=1}^{k+1} c a_i = \sum_{i=1}^{k} c a_i + c a_{k+1} \).
Exercise A.2.11 (continued 6)

Solution (continued). We have \( \sum_{i=1}^{k+1} ca_i \)

\[
\begin{align*}
&= \sum_{i=1}^{k} ca_i + ca_{k+1} \\
&= c \sum_{i=1}^{k} a_i + ca_{k+1} \text{ by the induction hypothesis} \\
&= c \left( \sum_{i=1}^{k} a_i + a_{k+1} \right) \text{ since multiplication distributes over addition in } \mathbb{R} \\
&= c \left( \sum_{i=1}^{k+1} a_i \right) = c \sum_{i=1}^{n} a_i \text{ where } n = k + 1,
\end{align*}
\]

so the result holds for \( n = k + 1 \) and, by the mathematical induction principle, the formula holds for all \( n \in \mathbb{N} \), as claimed.
Solution. First, we check the formula for \( n = 1 \). This gives \((c) = 1c\), which holds. Second, we assume the formula holds for \( n = k \), so that we assume \( \sum_{i=1}^{k} c = kc \). We want to show that the formula also holds for \( n = k + 1 \). Consider \( \sum_{i=1}^{k+1} c = \left( \sum_{i=1}^{k} c \right) + c \).
Constant Value Rule: \( \sum_{i=1}^{n} c = nc \)

**Solution.** First, we check the formula for \( n = 1 \). This gives \( (c) = 1c \), which holds. Second, we assume the formula holds for \( n = k \), so that we assume \( \sum_{i=1}^{k} c = kc \). We want to show that the formula also holds for \( n = k + 1 \). Consider \( \sum_{i=1}^{k+1} c = \left( \sum_{i=1}^{k} c \right) + c \).
Solution (continued). We have \( \sum_{i=1}^{k+1} c \)

\[
= \left( \sum_{i=1}^{k} c \right) + c
\]

\( = kc + c \) by the induction hypothesis

\( = (k + 1)c \) since multiplication distributes over addition in \( \mathbb{R} \)

\( = nc \) where \( n = k + 1 \),

so the result holds for \( n = k + 1 \) and, by the mathematical induction principle, the formula holds for all \( n \in \mathbb{N} \), as claimed. \( \square \)
Example A.2.B. Prove the General Product Rule (see Exercise 3.3.77 for motivation of this result): For differentiable functions $u_1, u_2, \ldots, u_n$, we have that the derivative of the product $u_1u_2\cdots u_n$ exists and

$$
\frac{d}{dx}[(u_1)(u_2)\cdots(u_n)] = [u_1'](u_2)(u_3)\cdots(u_{n-1})(u_n) \\
\quad + (u_1)[u_2'](u_3)\cdots(u_{n-1})(u_n) \\
\quad + (u_1)(u_2)[u_3']\cdots(u_{n-1})(u_n) + \cdots \\
\quad + (u_1)(u_2)(u_3)\cdots[u'_{n-1}](u_n) \\
\quad + (u_1)(u_2)(u_3)\cdots(u_{n-1})[u'_n].
$$
Example A.2.B (continued 1)

**Proof.** We introduce a product notation, similar to the summation notation: $u_1 u_2 \cdots u_n = \prod_{i=1}^{n} u_i$. We can then express the claim of this theorem as

$$\frac{d}{dx} \left[ \prod_{i=1}^{n} u_i \right] = \sum_{j=1}^{n} u'_j \prod_{i=1,i\neq j}^{n} u_i.$$

First, we check the formula for $n = 1$. This gives

$$\frac{d}{dx} [u_1] = \sum_{j=1}^{1} u'_j \prod_{i=1,i\neq 1}^{1} u_i = u'_1,$$

which holds. For clarity, we also check the formula for $n = 2$. This gives

$$\frac{d}{dx} [u_1 u_2] = \sum_{j=1}^{2} u'_j \prod_{i=1,i\neq j}^{2} u_i = [u'_1](u_2) + (u_1)[u'_2],$$

which holds by the Derivative Product Rule (Theorem 3.3.G).
Example A.2.B (continued 1)

Proof. We introduce a product notation, similar to the summation notation: \( u_1 u_2 \cdots u_n = \prod_{i=1}^{n} u_i \). We can then express the claim of this theorem as

\[
\frac{d}{dx} \left[ \prod_{i=1}^{n} u_i \right] = \sum_{j=1}^{n} u'_j \prod_{i=1, i \neq j}^{n} u_i.
\]

First, we check the formula for \( n = 1 \). This gives

\[
\frac{d}{dx} [u_1] = \sum_{j=1}^{1} u'_j \prod_{i=1, i \neq 1}^{1} u_i = u'_1, \text{ which holds.}
\]

For clarity, we also check the formula for \( n = 2 \). This gives

\[
\frac{d}{dx} [u_1 u_2] = \sum_{j=1}^{2} u'_j \prod_{i=1, i \neq j}^{2} u_i = [u'_1](u_2) + (u_1)[u'_2], \text{ which holds by the Derivative Product Rule (Theorem 3.3.G).}
\]
Example A.2.B (continued 2)

**Proof (continued).** Next, we assume the formula holds for \( n = k \), so that we assume 
\[
\frac{d}{dx} \left[ \prod_{i=1}^{k} u_i \right] = \sum_{j=1}^{k} u'_j \prod_{i=1, i \neq j}^{k} u_i. 
\]  
We want to show that the formula also holds for \( n = k + 1 \). Consider
\[
\frac{d}{dx} \left[ \prod_{i=1}^{k+1} u_i \right] = \frac{d}{dx} \left[ \left( \prod_{i=1}^{k} u_i \right) u_{k+1} \right]
\]
\[= \frac{d}{dx} \left[ \prod_{i=1}^{k} u_i \right] (u_{k+1}) + \left( \prod_{i=1}^{k} u_i \right) [u'_{k+1}] \text{ by the Derivative Rule for Products (Theorem 3.3.G)}
\]
\[= \sum_{j=1}^{k} u'_j \prod_{i=1, i \neq j}^{k} u_i (u_{k+1}) + \left( \prod_{i=1}^{k} u_i \right) [u'_{k+1}] \text{ by induction hypothesis}
\]
Example A.2.B (continued 2)

Proof (continued). Next, we assume the formula holds for \( n = k \), so that we assume

\[
\frac{d}{dx} \left[ \prod_{i=1}^{k} u_i \right] = \sum_{j=1}^{k} u_j' \prod_{i=1, i \neq j}^{k} u_i.
\]

We want to show that the formula also holds for \( n = k + 1 \). Consider

\[
\frac{d}{dx} \left[ \prod_{i=1}^{k+1} u_i \right] = \frac{d}{dx} \left[ \left( \prod_{i=1}^{k} u_i \right) u_{k+1} \right]
\]

\[
= \frac{d}{dx} \left( \prod_{i=1}^{k} u_i \right) (u_{k+1}) + \left( \prod_{i=1}^{k} u_i \right) [u'_{k+1}] \text{ by the Derivative Rule for Products (Theorem 3.3.G)}
\]

\[
= \left[ \sum_{j=1}^{k} u_j' \prod_{i=1, i \neq j}^{k} u_i \right] (u_{k+1}) + \left( \prod_{i=1}^{k} u_i \right) [u'_{k+1}] \text{ by induction hypothesis}
\]
Proof (continued). ... \[
\frac{d}{dx} \left[ \prod_{i=1}^{k+1} u_i \right] \\
= \left[ \sum_{j=1}^{k} u_j' \prod_{i=1, i \neq j}^{k} u_i \right] (u_{k+1}) + \left( \prod_{i=1}^{k} u_i \right) [u'_{k+1}] \\
= \sum_{j=1}^{k} u_j' \prod_{i=1, i \neq j}^{k+1} u_i + u'_{k+1} \prod_{i=1, i \neq k+1}^{k+1} u_i \\
= \sum_{j=1}^{k+1} u_j \prod_{i=1, i \neq j}^{k+1} u_i = \sum_{j=1}^{n} u_j \prod_{i=1, i \neq j}^{n} u_i \text{ where } n = k + 1,
\]

so the result holds for \( n = k + 1 \) and, by the mathematical induction principle, the formula holds for all \( n \in \mathbb{N} \), as claimed.