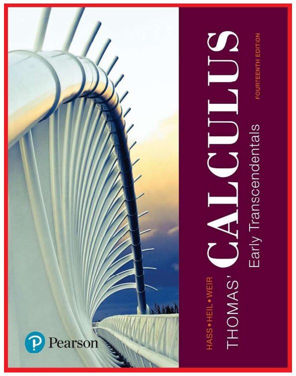
Calculus 1

Appendices

A.4. Proofs of Limit Theorems—Examples and Proofs



Theorem 2.1(4)

Theorem 2.1(4). Limit Product Rule.
If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

$$\lim_{x \to c} (f(x)g(x)) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right) = LM.$$

Proof. Let $\varepsilon > 0$. Since $\lim_{x \to c} f(x) = L$ and $\sqrt{\varepsilon/3} > 0$, then there exists $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$ then $|f(x) - L| < \sqrt{\varepsilon/3}$. Since $\lim_{x \to c} f(x) = L$ and $\varepsilon/(3(1 + |M|)) > 0$, then there exists $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$ then $|f(x) - L| < \varepsilon/(3(1 + |M|))$. Since $\lim_{x \to c} g(x) = M$ and $\sqrt{\varepsilon/3} > 0$, then there exists $\delta_3 > 0$ such that if $0 < |x - c| < \delta_3$ then $|g(x) - M| < \sqrt{\varepsilon/3}$. Since $\lim_{x \to c} g(x) = M$ and $\varepsilon/(3(1 + |L|) > 0$, then there exists $\delta_4 > 0$ such that if $0 < |x - c| < \delta_4$ then $|g(x) - M| < \varepsilon/(3(1 + |L|))$. Choose $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$.

Theorem 2.1(4) (continued 1)

Proof (continued). If $0 < |x - c| < \delta \le \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$, then

$$\begin{aligned}
f(x)g(x) - LM &= (f(x)g(x) - LM) + 2(LM - LM) \\
&\quad + (Lg(x) - Lg(x)) + (Mf(x) - Mf(x)) \\
&= LM + Lg(x) - LM + Mf(x) + f(x)g(x) \\
&\quad - Mf(x) - LM - Lg(x) + LM - LM \\
&= (LM + Mf(x) - LM) + (Lg(x) + f(x)g(x) \\
&\quad - Lg(x)) - (LM + Mf(x) - LM) - LM \\
&= (L + (f(x) - L))M + (L + (f(x) - L))g(x) \\
&\quad - (L + (f(x) - L))M - LM \\
&= (L + (f(x) - L))(M + (g(x) - M)) - LM \\
&= LM + L(g(x) - M)) + M(f(x) - L) \\
&\quad + (f(x) - L)(g(x) - M) - LM \\
&= L(g(x) - M) + M(f(x) - L) + (f(x) - L)(g(x) - M).
\end{aligned}$$

Theorem 2.1(4) (continued 2)

Proof (continued). Since
$f(x)g(x) - LM = L(g(x) - M) + M(f(x) - L) + (f(x) - L)(g(x) - M)$, then by the Triangle Inequality for absolute value (see Exercise A.1.24), we have

$$\begin{aligned}
|f(x)g(x) - LM| &\le |L(g(x) - M)| + |M(f(x) - L)| \\
&\quad + |f(x) - L|\,|g(x) - M| \\
&= |L|\,|g(x) - M| + |M|\,|f(x) - L| \\
&\quad + |f(x) - L|\,|g(x) - M| \\
&< (1 + |L|)|g(x) - M| + (1 + |M|)|f(x) - L| \\
&\quad + |f(x) - L|\,|g(x) - M| \text{ since } |L| < 1 + |L| \\
&\quad \text{ and } |M| < 1 + |M| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sqrt{\frac{\varepsilon}{3}}\sqrt{\frac{\varepsilon}{3}} \text{ since } 0 < |x - c| < \delta \\
&= \varepsilon.
\end{aligned}$$

Theorem 2.1(4) (continued 3)

Theorem 2.1(4). Limit Product Rule.
If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

$$\lim_{x\to c}(f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to x} g(x)\right) = LM.$$

Proof (continued). That is, if $0 < |x - c| < \delta$ then $|f(x)g(x) - LM| < \varepsilon$. Therefore, by the definition of limit,

$$\lim_{x\to c}(f(x)g(x)) = LM,$$

as claimed. □

Theorem 2.1(5)

Theorem 2.1(5). Limit Quotient Rule.
If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{M},$$

if $\lim_{x\to c} g(x) = M \neq 0$.

Proof. First, we show that $\lim_{x\to c} 1/g(x) = 1/M$. Let $\varepsilon > 0$. Since $\lim_{x\to c} g(x) = M$ by hypothesis, then there exists $\delta_1 > 0$ such that

$$\text{if } 0 < |x - c| < \delta_1 \text{ then } |g(x) - M| < |M|/2. \qquad (*)$$

By the Triangle Inequality, $|A + B| \le |A| + |B|$ for all $A, B \in \mathbb{R}$ (see A.1. Real Numbers and the Real Line and Exercise A.1.24). So $|A| = |(A - B) + B| \le |A - B| + |B|$ or $|A| - |B| \le |A - B|$, and $|B| = |(A - B) - A| \le |A - B| + |-A| = |A - B| + |A|$ or $|B| - |A| \le |A - B|$. Therefore $||A| - |B|| \le |A - B|$.

Theorem 2.1(5) (continued 1)

Proof (continued). Therefore $||A| - |B|| \le |A - B|$. So with $A = g(x)$ and $B = M$, we have $||g(x)| - |M|| \le |g(x) - M|$. So if $0 < |x - c| < \delta_1$ then $||g(x)| - |M|| \le |g(x) - M| < |M|/2$ by $(*)$. This implies $-|M|/2 < |g(x)| - |M| < |M|/2$ or $|M|/2 < |g(x)| < 3|M|/2$ or $|M| < 2|g(x)| < 3|M|$. Now $|M| < 2|g(x)|$ gives $1/|g(x)| < 2/|M|$, and $2|g(x)| < 3|M|$ or $|g(x)|/3 < |M|/2$ gives $2/|M| < 3/|g(x)|$. Therefore, if $0 < |x - c| < \delta_1$ then

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M - g(x)}{Mg(x)}\right|$$

$$= \frac{1}{|M|}\frac{1}{|g(x)|}|M - g(x)| < \frac{1}{|M|}\frac{2}{|M|}|M - g(x)|, \qquad (5)$$

where the last inequality holds since $1/|g(x)| < 1/|M|$ for $0 < |x - c| < \delta_1$.

Theorem 2.1(5) (continued 2)

Proof (continued). Now $(1/2)|M|^2\varepsilon > 0$, so there exists $\delta_2 > 0$ such that

$$\text{if } 0 < |x - c| < \delta_2 \text{ then } |M - g(x)| < \varepsilon|M|^2/2. \qquad (6)$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $0 < |x - c| < \delta$, we have both $\left|\frac{1}{g(x)} - \frac{1}{M}\right| < \frac{1}{|M|}\frac{2}{|M|}|M - g(x)|$ by (5) and $|M - g(x)| < \varepsilon|M|^2/2$ by (6). So for such x,

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| < \frac{1}{|M|}\frac{2}{|M|}|M - g(x)| < \frac{1}{|M|}\frac{2}{|M|}\frac{\varepsilon|M|^2}{2} = \varepsilon.$$

So $\lim_{x\to c} 1/g(x) = 1/M$ by the definition of limit.
Finally, by the Limit Product Rule (Theorem 2.1.(4))

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c} f(x)\lim_{x\to c}\frac{1}{g(x)} = L\frac{1}{M} = \frac{L}{M},$$

as claimed. □

Theorem 2.4

Theorem 2.4. Sandwich Theorem.
Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval I containing c, except possibly at $x = c$ itself. Suppose also that $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. Then $\lim\limits_{x\to c} f(x) = L$.

Solution. We give proofs for right-hand and left-hand one-sided limits. Let $\varepsilon > 0$.

Suppose $\lim_{x\to c^+} g(x) = \lim_{x\to c^+} h(x) = L$. Then there exists $\delta_1 > 0$ such that if $c < x < c + \delta_1$ and $x \in I$ then $|g(x) - L| < \varepsilon$. There also exists $\delta_2 > 0$ such that if $c < x < c + \delta_2$ and $x \in I$ then $|h(x) - L| < \varepsilon$. With $\delta = \min\{\delta_1, \delta_2\}$, we have that if $c < x < c + \delta$ and $x \in I$ then both $|g(x) - L| < \varepsilon$ and $|h(x) - L| < \varepsilon$; that is, both $L - \varepsilon < g(x) < L + \varepsilon$ and $L - \varepsilon < h(x) < L + \varepsilon$. So if $c < x < c + \delta$ and $x \in I$, then $L - \varepsilon < g(x) \leq f(x) \leq h(x) < L + \varepsilon$, and so $L - \varepsilon < f(x) < L + \varepsilon$ or $|f(x) - L| < \varepsilon$. Therefore, by the definition of limit, $\lim_{x\to c^+} f(x) = L$.

Theorem 2.4. Sandwich Theorem.

Theorem 2.4 (continued)

Solution (continued). Suppose $\lim_{x\to c^-} g(x) = \lim_{x\to c^-} h(x) = L$. Then there exists $\delta_2 > 0$ such that if $c - \delta_2 < x < c$ and $x \in I$ then $|g(x) - L| < \varepsilon$. There also exists $\delta_2 > 0$ such that if $c - \delta_2 < x < c$ and $x \in I$ then $|h(x) - L| < \varepsilon$. With $\delta = \min\{\delta_1, \delta_2\}$, we have that if $c - \delta < x < c$ and $x \in I$ then both $|g(x) - L| < \varepsilon$ and $|h(x) - L| < \varepsilon$; that is, both $L - \varepsilon < g(x) < L + \varepsilon$ and $L - \varepsilon < h(x) < L + \varepsilon$. So if $c - \delta < x < c$ and $x \in I$, then $L - \varepsilon < g(x) \leq f(x) \leq h(x) < L + \varepsilon$, and so $L - \varepsilon < f(x) < L + \varepsilon$ or $|f(x) - L| < \varepsilon$. Therefore, by the definition of limit, $\lim_{x\to c^-} f(x) = L$.

If $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then by the above results for one sided limits we have $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$ and by Theorem 2.6 (Relation Between One-Sided and Two-Sided Limits) we have $\lim_{x\to c} f(x) = L$, as claimed. □