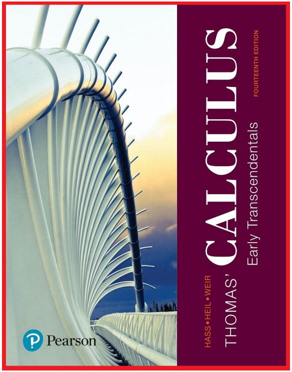
Calculus 1

Chapter 1. Functions

1.1. Functions and Their Graphs—Examples and Proofs



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Exercise 1.1.4. Find the domain and range of $g(x) = \sqrt{x^2 - 3x}$.

Solution. We cannot take square roots of negative numbers, so we need $x^2 - 3x \geq 0$ or $x(x-3) \geq 0$. Now $x^2 - 3x = x(x-3) = 0$ for $x = 0$ and $x = 3$. Since the graph of $y = x^2 - 3x$ is a parabola, then it represents a continuous function, and the only way it can change sign (from positive to negative, or from negative to positive) is by passing through the value 0. So *if* $x^2 - 3x = x(x-3)$ changes sign, then it does it at $x = 0$ or $x = 3$ and so $x^2 - 3x = x(x-3)$ has the same sign on the intervals $(-\infty, 0)$, $(0, 3)$, and $(3, \infty)$. We just need to test the sign of $x^2 - 3x$ at a test value from each interval.

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interval	test value k	$f(k) = k^2 - 3k$	Sign of $f(x)$
$(-\infty, 0)$	-1	$(-1)^2 - 3(-1) = 4$	$+$
$(0, 3)$	1	$(1)^2 - 3(1) = -2$	$-$
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Exercise 1.1.4 (continued)

Solution (continued). We see that $x^2 - 3x > 0$ for $x \in (-\infty, 0) \cup (3, \infty)$, and so $x^2 - 3x \geq 0$ for $x \in (-\infty, 0] \cup [3, \infty)$. That is, the $\boxed{\text{domain of } g \text{ is } (-\infty, 0] \cup [3, \infty)}$.

For r in the range of g (that is, if $r = \sqrt{x^2 - 3x}$ for some x in the domain of g), we must have $r \geq 0$ (since square roots are never negative; see Appendix A1. Real Numbers and the Real Line). For such r, if $r = g(x) = \sqrt{x^2 - 3x}$ then $r^2 = (\sqrt{x^2 - 3x})^2 = x^2 - 3x$ or $x^2 - 3x - r^2 = 0$ and by the quadratic equation, $x = \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-r^2)}}{2(1)} = \dfrac{3 \pm \sqrt{9 + 4r^2}}{2}$. Since $9 + 4r^2 \geq 0$ then such an x exists and so r is in the range of g, provided $r \geq 0$. That is, the $\boxed{\text{range of } g \text{ is } [0, \infty)}$. □

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Notice that $\dfrac{3 - \sqrt{9 + 4r^2}}{2} \leq 0$ and $\dfrac{3 + \sqrt{9 + 4r^2}}{2} \geq 3$. So there are two x values that produce output value $r \geq 0$ (one in $(-\infty, 0)$ and one in $(3, \infty)$).

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$$g(x) = \begin{cases} 1 - x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2. \end{cases}$$

Solution. The graph of $y = 1 - x$ is a line of slope $m_1 = -1$ containing the point $(0, 1)$ (this is the y-intercept). The graph of $y = 2 - x$ is a line of slope $m_2 = -1$ containing the point $(2, 0)$ (this is the x-intercept). So we graph $y = 1 - x$ for $0 \leq x \leq 1$ and $y = 2 - x$ for $1 < x \leq 2$:

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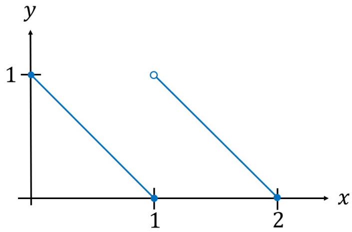

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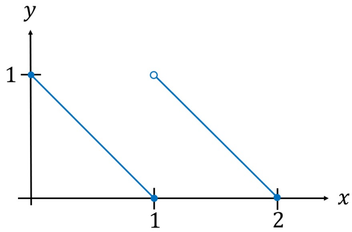

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Exercise 1.1.58

Exercise 1.1.58. Determine whether the function $h(t) = 2|t| + 1$ is even, odd, or neither.

Solution. We replace the independent variable t with $-t$ to test for evenness or oddness:

$$h(-t) = 2|(-t)| + 1 = 2|-1|\,|t| + 1 = 2(1)|t| + 1 = 2|t| + 1 = h(t).$$

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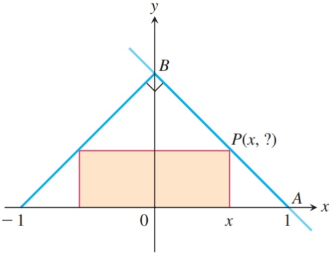

Proof. **(a)** Let O represent the origin. The y-axis bisects the right angle in the given isosceles triangle, so triangle AOB is similar to the given isosceles triangle and therefore is itself an isosceles triangle. That is, the length of segment OB is 1 and so point B is $(0, 1)$.

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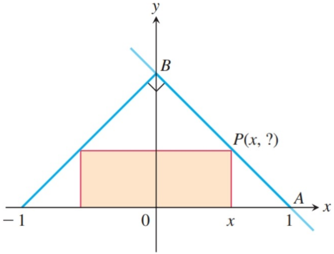

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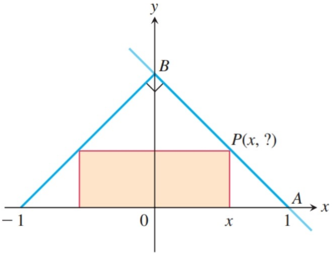
(a) Express the y-coordinate of P in terms of x.



Proof (continued). So the equation of the line passing through $A = (x_1, y_1) = (1, 0)$ and $B = (x_2, y_2) = (0, 1)$ has slope $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{(1) - (0)}{(0) - (1)} = -1$ and hence is, by the point slope formula, $y - y_1 = m(x - x_1)$ or $y - (0) = (-1)(x - 1)$ or $y = -x + 1$. So point P is of the form $(x, y) = (x, -x + 1)$. That is, the y-coordinate of P in terms of x is $\boxed{-x + 1}$.

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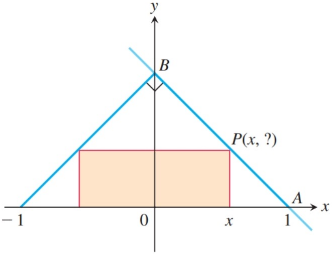

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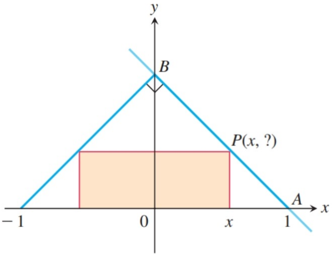
(b) Express the area of the rectangle in terms of x.



Proof (continued). **(b)** Since point P is of the form $(x, y) = (x, -x + 1)$, the width of the rectangle is $2x$, and the height of the rectangle is y, then the area of the rectangle is $A = 2xy$ or $\boxed{A = 2x(-x + 1)}$. □

Notice that the values given in (a) and (b) are only (physically) meaningful for $x \in [0, 1]$.

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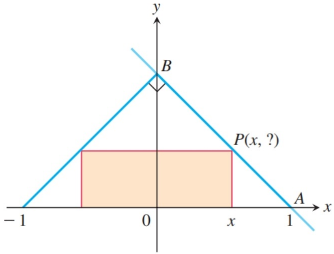
(b) Express the area of the rectangle in terms of x.



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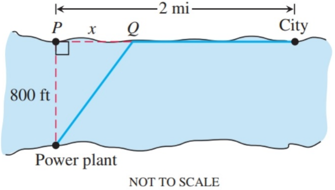
A power plant sits next to a river where the river is 800 ft wide. To lay a new cable from the plant to a location in the city 2 mi downstream on the opposite side costs \$180 per foot across the river and \$100 per foot along the land.



(a) Suppose that the cable goes from the plant to a point Q on the opposite side that is x ft from the point P directly opposite the plant. Write a function $C(x)$ that gives the cost of laying the cable in terms of the distance x.

(b) Generate a table of values to determine if the least expensive location for point Q is less than 2000 ft or greater than 2000 ft from point P.

Solution. (a) First, 2 mile = (2 mile)(5,280 ft/mile) = 10,560 ft. We see that there is $(10{,}560 - x)$ ft of cable along the land.

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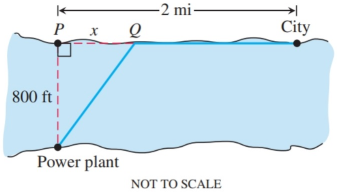

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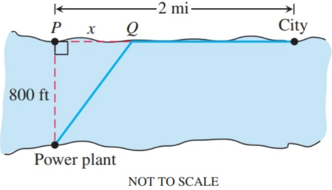

Solution (continued). Since the Power Plant, point P, and point Q form a right triangle, then by the Pythagorean Theorem we see that the amount of cable across the river is $\sqrt{(800)^2 + x^2}$ ft. Since the cable costs \$180 per foot across the river and there is $\sqrt{(800)^2 + x^2}$ ft of cable across the river, then the cost of this part of the cable is $\$180\sqrt{(800)^2 + x^2}$. Since the cable costs \$100 per foot along the land and there is $(10{,}560 - x)$ ft of cable along the land, then the cost of this part of the cable is $\$100(10{,}560 - x)$. So the total cost of the cable is

$$\boxed{C(x) = 180\sqrt{(800)^2 + x^2} + 100(10{,}560 - x) \text{ dollars}}.$$

Exercise 1.1.76 (continued 2)

(b) Generate a table of values to determine if the least expensive location for point Q is less than 2000 ft or greater than 2000 ft from point P.

Solution (continued). We make a table of values of $C(x) = 180\sqrt{(800)^2 + x^2} + 100(10{,}560 - x)$ dollars:

x	$C(x)$
2300	\$1,264,328.64
2200	\$1,257,369.20
2100	\$1,250,499.69
2000	\$1,243,731.87
1900	\$1,237,079.51
1800	\$1,230,558.88
1700	\$1,224,289.30

It appears that the least expensive location for point Q is **less than 2000 ft** from point P. □