Calculus 1

Chapter 1. Functions

1.3. Trigonometric Functions—Examples and Proofs

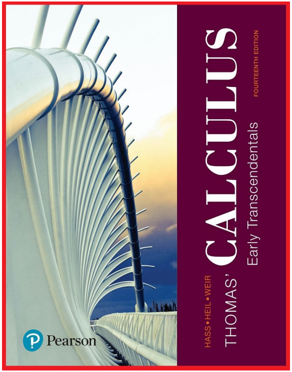


Table of contents

Exercise 1.3.2

Exercise 1.3.2. A central angle in a circle of radius 8 is subtended by an arc of length 10π. Find the angle's radian and degree measure.

Solution. The radius is $r = 8$ and the arc length is $s = 10\pi$. Since $\theta = s/r$, then here $\theta = (10\pi)/8 = \boxed{5\pi/4}$.

Exercise 1.3.2

Exercise 1.3.2. A central angle in a circle of radius 8 is subtended by an arc of length 10π. Find the angle's radian and degree measure.

Solution. The radius is $r = 8$ and the arc length is $s = 10\pi$. Since $\theta = s/r$, then here $\theta = (10\pi)/8 = \boxed{5\pi/4}$.

To convert θ to degrees, we multiply by the conversion factor of $180^\circ/\pi$ (or, if you like, $(180/\pi)^\circ$/radian; but remember that that radians are unitless). So we have $\theta = (5\pi/4)(180^\circ/\pi) = \boxed{225^\circ}$. □

Exercise 1.3.2

Exercise 1.3.2. A central angle in a circle of radius 8 is subtended by an arc of length 10π. Find the angle's radian and degree measure.

Solution. The radius is $r = 8$ and the arc length is $s = 10\pi$. Since $\theta = s/r$, then here $\theta = (10\pi)/8 = \boxed{5\pi/4}$.

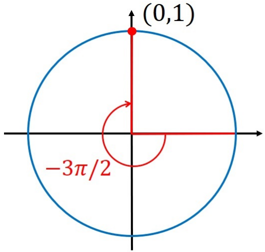
To convert θ to degrees, we multiply by the conversion factor of $180^\circ/\pi$ (or, if you like, $(180/\pi)^\circ$/radian; but remember that that radians are unitless). So we have $\theta = (5\pi/4)(180^\circ/\pi) = \boxed{225^\circ}$. □

Exercise 1.3.6

Exercise 1.3.6. Finish the following table of trigonometric values of some special angles:

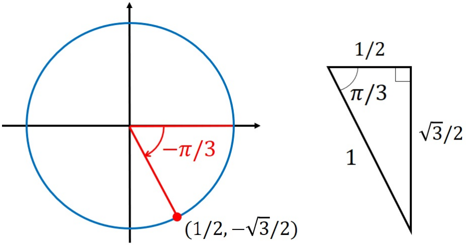
θ	$-3\pi/2$	$-\pi/3$	$-\pi/6$	$\pi/4$	$5\pi/6$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

Exercise 1.3.6 (continued 1)



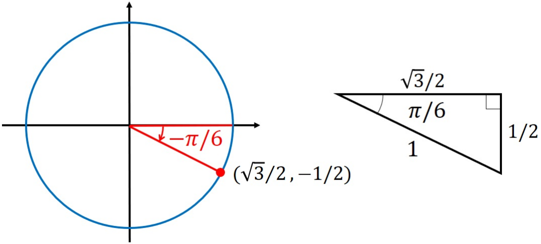
Solution. For $\theta = -3\pi/2$, the point on the unit circle and terminal side of θ is $(x, y) = (0, 1)$. By definition, since $r = 1$ on the unit circle, we have $\sin(-3\pi/2) = y/r = 1/1 = 1$, $\cos(-3\pi/2) = x/r = 0/1 = 0$, $\sec(-3\pi/2) = r/x$ is undefined, $\csc(-3\pi/2) = r/y = 1/1 = 1$, $\tan(-3\pi/2) = y/x$ is undefined, and $\cot(-3\pi/2) = x/y = 0/1 = 0$.

Exercise 1.3.6 (continued 2)



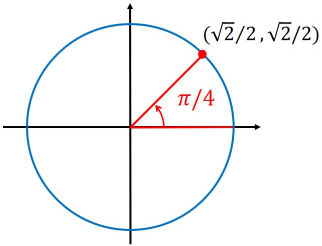
Solution (continued). For $\theta = -\pi/3$, we use the special right triangle containing an angle of $\pi/3$ to find that the point on the unit circle and terminal side of θ is $(x, y) = (1/2, -\sqrt{3}/2)$. By definition, since $r = 1$ on the unit circle, we have $\sin(-\pi/3) = y/r = (-\sqrt{3}/2)/(1) = -\sqrt{3}/2$, $\cos(-\pi/3) = x/r = (1/2)/(1) = 1/2$, $\sec(-\pi/3) = r/x = (1)/(1/2) = 2$, $\csc(-\pi/3) = r/y = (1)/(-\sqrt{3}/2) = -2/\sqrt{3}$, $\tan(-\pi/3) = y/x = (-\sqrt{3}/2)/(1/2) = -\sqrt{3}$, and $\cot(-\pi/3) = x/y = (1/2)/(-\sqrt{3}/2) = -1/\sqrt{3}$.

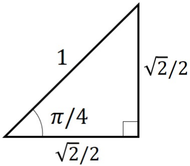
Exercise 1.3.6 (continued 3)



Solution (continued). For $\theta = -\pi/6$, we use the special right triangle containing an angle of $\pi/6$ to find that the point on the unit circle and terminal side of θ is $(x, y) = (\sqrt{3}/2, -1/2)$. By definition, since $r = 1$ on the unit circle, we have $\sin(-\pi/6) = y/r = (-1/2)/(1) = -1/2$,
$\cos(-\pi/6) = x/r = (\sqrt{3}/2)/(1) = \sqrt{3}/2$,
$\sec(-\pi/6) = r/x = (1)/(\sqrt{3}/2) = 2/\sqrt{3}$,
$\csc(-\pi/6) = r/y = (1)/(-1/2) = -2$,
$\tan(-\pi/6) = y/x = (-1/2)/(\sqrt{3}/2) = -1/\sqrt{3}$, and
$\cot(-\pi/6) = x/y = (\sqrt{3}/2)/(-1/2) = -\sqrt{3}$.

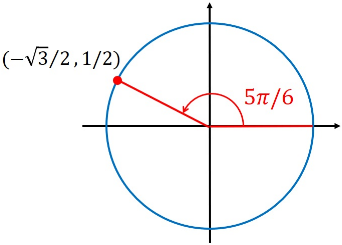
Exercise 1.3.6 (continued 4)

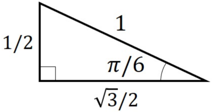




Solution (continued). For $\theta = \pi/4$, we use the special right triangle containing an angle of $\pi/4$ to find that the point on the unit circle and terminal side of θ is $(x, y) = (\sqrt{2}/2, \sqrt{2}/2)$. By definition, since $r = 1$ on the unit circle, we have $\sin(\pi/4) = y/r = (\sqrt{2}/2)/(1) = \sqrt{2}/2$,
$\cos(\pi/4) = x/r = (\sqrt{2}/2)/(1) = \sqrt{2}/2$,
$\sec(\pi/4) = r/x = (1)/(\sqrt{2}/2) = \sqrt{2}$,
$\csc(\pi/4) = r/y = (1)/(\sqrt{2}/2) = \sqrt{2}$,
$\tan(\pi/4) = y/x = (\sqrt{2}/2)/(\sqrt{2}/2) = 1$, and
$\cot(\pi/4) = x/y = (\sqrt{2}/2)/(\sqrt{2}/2) = 1$.

Exercise 1.3.6 (continued 5)





Solution (continued). For $\theta = 5\pi/6$, we use the special right triangle containing an angle of $5\pi/6$ to find that the point on the unit circle and terminal side of θ is $(x, y) = (-\sqrt{3}/2, 1/2)$. By definition, since $r = 1$ on the unit circle, we have $\sin(5\pi/6) = y/r = (1/2)/(1) = 1/2$,
$\cos(5\pi/6) = x/r = (-\sqrt{3}/2)/(1) = -\sqrt{3}/2$,
$\sec(5\pi/6) = r/x = (1)/(-\sqrt{3}/2) = -2/\sqrt{3}$,
$\csc(5\pi/6) = r/y = (1)/(1/2) = 2$,
$\tan(5\pi/6) = y/x = (1/2)/(-\sqrt{3}/2) = -1/\sqrt{3}$, and
$\cot(5\pi/6) = x/y = (-\sqrt{3}/2)/(1/2) = -\sqrt{3}$.

Exercise 1.3.6 (continued 6)

Solution (continued). We therefore have:

θ	$-3\pi/2$	$-\pi/3$	$-\pi/6$	$\pi/4$	$5\pi/6$
$\sin\theta$	1	$-\sqrt{3}/2$	$-1/2$	$\sqrt{2}/2$	$1/2$
$\cos\theta$	0	$1/2$	$\sqrt{3}/2$	$\sqrt{2}/2$	$-\sqrt{3}/2$
$\tan\theta$	UND	$-\sqrt{3}$	$-1/\sqrt{3}$	1	$-1/\sqrt{3}$
$\cot\theta$	0	$-1/\sqrt{3}$	$-\sqrt{3}$	1	$-\sqrt{3}$
$\sec\theta$	UND	2	$2/\sqrt{3}$	$\sqrt{2}$	$-2/\sqrt{3}$
$\csc\theta$	1	$-2/\sqrt{3}$	-2	$\sqrt{2}$	2

□

Exercise 1.3.31

Exercise 1.3.31. Use the addition formulas to derive the identity $\cos\left(x - \dfrac{\pi}{2}\right) = \sin x$.

Solution. We have the formula $\cos(A - B) = \cos A \cos B + \sin A \sin B$, so with $A = x$ and $B = \pi/2$ we have
$\cos(x - \pi/2) = \cos x \cos \pi/2 + \sin x \sin \pi/2 = \cos x(0) + \sin x(1) = \sin x$.
□

Exercise 1.3.31

Exercise 1.3.31. Use the addition formulas to derive the identity $\cos\left(x - \frac{\pi}{2}\right) = \sin x$.

Solution. We have the formula $\cos(A - B) = \cos A \cos B + \sin A \sin B$, so with $A = x$ and $B = \pi/2$ we have $\cos(x - \pi/2) = \cos x \cos \pi/2 + \sin x \sin \pi/2 = \cos x(0) + \sin x(1) = \sin x$. □

Notice that x and $x - \pi/2$ are complementary angles since $(x) + (x - \pi/2) = \pi/2$. So this exercise shows that the sine of an angle equals the **co**sine of its **co**mplement; *this* is why cosine is called "**co**sine."

Exercise 1.3.31

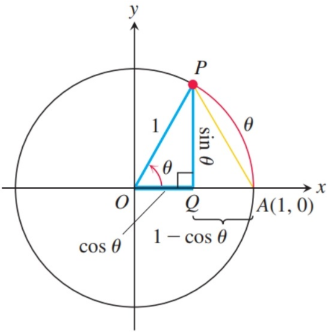
Exercise 1.3.31. Use the addition formulas to derive the identity $\cos\left(x - \frac{\pi}{2}\right) = \sin x$.

Solution. We have the formula $\cos(A - B) = \cos A \cos B + \sin A \sin B$, so with $A = x$ and $B = \pi/2$ we have
$\cos(x - \pi/2) = \cos x \cos \pi/2 + \sin x \sin \pi/2 = \cos x(0) + \sin x(1) = \sin x$.
□

Notice that x and $x - \pi/2$ are complementary angles since $(x) + (x - \pi/2) = \pi/2$. So this exercise shows that the sine of an angle equals the **co**sine of its **co**mplement; *this* is why cosine is called "**co**sine."

Example 1.3.A

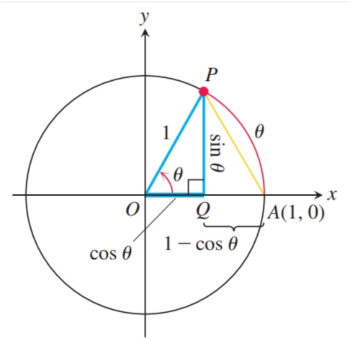
Example 1.3.A. For any angle θ measured in radians, we have $-|\theta| \leq \sin\theta \leq |\theta|$ and $-|\theta| \leq 1 - \cos\theta \leq |\theta|$.



Solution. As in Figure 1.47, we put θ in standard position. Since the circle is a unit circle (that is, $r = 1$), then $|\theta|$ equals the length of the circular arc AP.

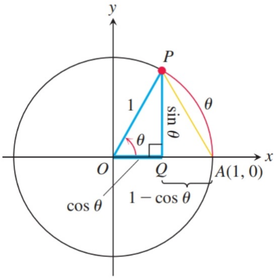
Example 1.3.A

Example 1.3.A. For any angle θ measured in radians, we have $-|\theta| \leq \sin\theta \leq |\theta|$ and $-|\theta| \leq 1 - \cos\theta \leq |\theta|$.



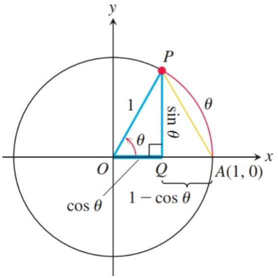
Solution. As in Figure 1.47, we put θ in standard position. Since the circle is a unit circle (that is, $r = 1$), then $|\theta|$ equals the length of the circular arc AP.

Example 1.3.A (continued)



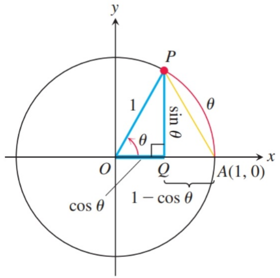
Solution (continued). We see from the figure that the length of line segment AP is less than or equal to $|\theta|$. Triangle APQ is a right triangle with sides of length $QP = |\sin\theta|$ and $AQ = 1 - \cos\theta$. So by the Pythagorean Theorem (and the fact that $AP \le |\theta|$) we have $\sin^2\theta + (1 - \cos\theta)^2 = (AP)^2 \le \theta^2$. So we have both $\sin^2\theta \le \theta^2$ and $(1 - \cos\theta)^2 \le \theta^2$.

Example 1.3.A (continued)



Solution (continued). We see from the figure that the length of line segment AP is less than or equal to $|\theta|$. Triangle APQ is a right triangle with sides of length $QP = |\sin\theta|$ and $AQ = 1 - \cos\theta$. So by the Pythagorean Theorem (and the fact that $AP \le |\theta|$) we have $\sin^2\theta + (1-\cos\theta)^2 = (AP)^2 \le \theta^2$. So we have both $\sin^2\theta \le \theta^2$ and $(1-\cos\theta)^2 \le \theta^2$. Taking square roots (and observing that the square root function is an increasing function so that it preserves inequalities), $\sqrt{\sin^2\theta} \le \sqrt{\theta^2}$ and $\sqrt{(1-\cos\theta)^2} \le \sqrt{\theta^2}$, or $|\sin\theta| \le |\theta|$ and $|1-\cos\theta| \le |\theta|$. These two inequalities imply that $-|\theta| \le \sin\theta \le |\theta|$ and $-|\theta| \le 1-\cos\theta \le |\theta|$, as claimed (see Appendix A.1. Real Numbers and the Real Line where intervals are related to absolute values). □

Example 1.3.A (continued)



Solution (continued). We see from the figure that the length of line segment AP is less than or equal to $|\theta|$. Triangle APQ is a right triangle with sides of length $QP = |\sin\theta|$ and $AQ = 1 - \cos\theta$. So by the Pythagorean Theorem (and the fact that $AP \le |\theta|$) we have $\sin^2\theta + (1-\cos\theta)^2 = (AP)^2 \le \theta^2$. So we have both $\sin^2\theta \le \theta^2$ and $(1-\cos\theta)^2 \le \theta^2$. Taking square roots (and observing that the square root function is an increasing function so that it preserves inequalities), $\sqrt{\sin^2\theta} \le \sqrt{\theta^2}$ and $\sqrt{(1-\cos\theta)^2} \le \sqrt{\theta^2}$, or $|\sin\theta| \le |\theta|$ and $|1-\cos\theta| \le |\theta|$. These two inequalities imply that $-|\theta| \le \sin\theta \le |\theta|$ and $-|\theta| \le 1-\cos\theta \le |\theta|$, as claimed (see Appendix A.1. Real Numbers and the Real Line where intervals are related to absolute values). □

Exercise 1.3.68

Exercise 1.3.68. The general sine curve is

$$f(x) = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D.$$

For $y = \frac{1}{2}\sin(\pi x - \pi) + \frac{1}{2}$ identify A, B, C, and D and sketch the graph.

Solution. First we write
$y = \frac{1}{2}\sin(\pi x - \pi) + \frac{1}{2} = \frac{1}{2}\sin(\pi(x-1)) + \frac{1}{2} = \frac{1}{2}\sin\left(\frac{2\pi}{2}(x-1)\right) + \frac{1}{2}$.
We have $A = 1/2$, $B = 2$, $C = 1$, and $D = 1/2$. Now A is the amplitude, B is the period, C is the horizontal shift, and $y = D$ is the axis. ...

Exercise 1.3.68

Exercise 1.3.68. The general sine curve is

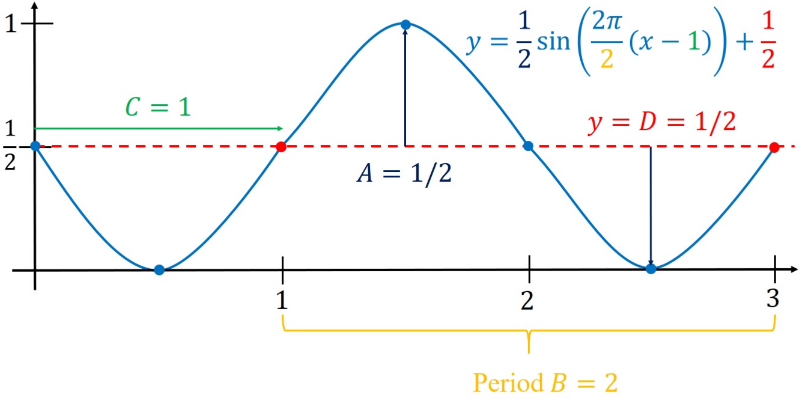
$$f(x) = A\sin\left(\frac{2\pi}{B}(x-C)\right) + D.$$

For $y = \frac{1}{2}\sin(\pi x - \pi) + \frac{1}{2}$ identify A, B, C, and D and sketch the graph.

Solution. First we write
$y = \frac{1}{2}\sin(\pi x - \pi) + \frac{1}{2} = \frac{1}{2}\sin(\pi(x-1)) + \frac{1}{2} = \frac{1}{2}\sin\left(\frac{2\pi}{2}(x-1)\right) + \frac{1}{2}.$
We have $A = 1/2$, $B = 2$, $C = 1$, and $D = 1/2$. Now A is the amplitude, B is the period, C is the horizontal shift, and $y = D$ is the axis. ...

Exercise 1.3.68 (continued)

Solution (continued). We have $A = 1/2$, $B = 2$, $C = 1$, and $D = 1/2$. Now A is the amplitude, B is the period, C is the horizontal shift, and $y = D$ is the axis.



□