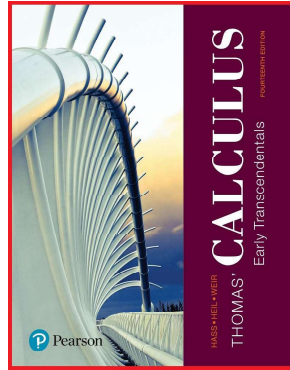


Calculus 1

Chapter 1. Functions

1.6. Inverse Functions and Logarithms—Examples and Proofs

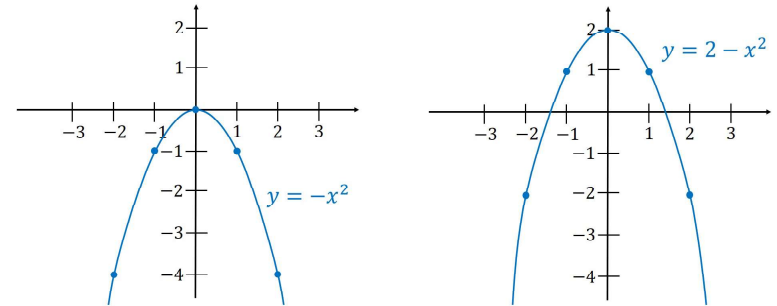


Exercise 1.6.10

Exercise 1.6.10. Determine from its graph if the function

$$f(x) = \begin{cases} 2 - x^2, & x \leq 1 \\ x^2, & x > 1 \end{cases} \text{ is one-to-one.}$$

Solution. The pieces of f are translations and reflections of the parabola $y = x^2$. The graph of $y = 2 - x^2$ is the reflection of the parabola $y = x^2$ about the x axis (to produce $y = -x^2$) which is then translated up by 2 units.

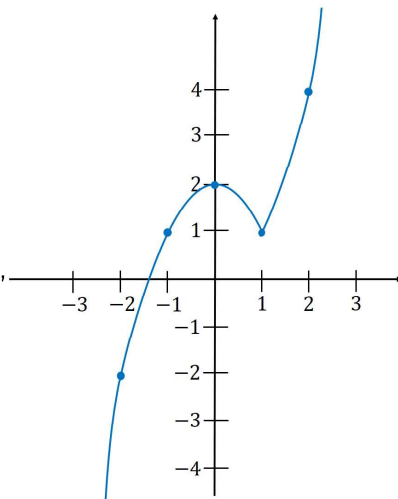


Exercise 1.6.10 (continued)

Solution (continued).

So we graph $y = 2 - x^2$ for $x \leq 1$ and $y = x^2$ for $x > 1$:

We see from the graph of $y = f(x)$ that it is not one-to-one because, for example, the value 1 is attained at two x -values (namely, $x = -1$ and $x = 1$), and the value 2 is attained at two x -values (namely, $x = 0$ and $x = \sqrt{2}$). In addition, each value in $(1, 2)$ is attained three times! \square



Example 1.6.4

Example 1.6.4. Find the inverse of the function $y = x^2$, $x \geq 0$. See Figure 1.59.

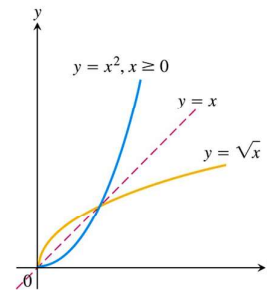


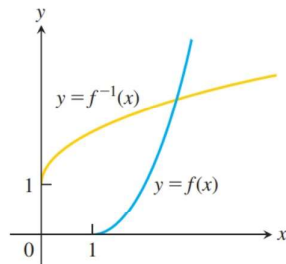
Figure 1.59

Solution. We follow the two step procedure. First, let $y = x^2$ where $x \geq 0$. Solving for x we have $\sqrt{y} = \sqrt{x^2}$ or $\sqrt{y} = |x|$ where $x \geq 0$. Since $|x| = x$ for $x \geq 0$, then $\sqrt{y} = x$.

Interchanging x and y gives $\sqrt{x} = y$, so that $y = f^{-1}(x) = \sqrt{x}$. \square

Exercise 1.6.22

Exercise 1.6.22. Here is a graph of $f(x) = x^2 - 2x + 1$, $x \geq 1$, and its inverse. Find a formula for f^{-1} .



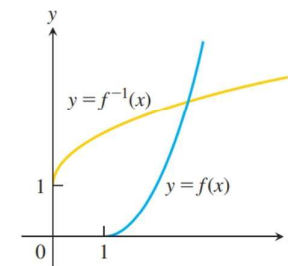
Solution. We follow the two step procedure. First, let $y = x^2 - 2x + 1$ where $x \geq 1$. Then $x^2 - 2x + (1 - y) = 0$ and solving for x we have by the quadratic equation that

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1-y)}}{2(1)} = \frac{2 \pm \sqrt{4y}}{2} = \frac{2 \pm 2\sqrt{y}}{2} = 1 \pm \sqrt{y}$$

where $x \geq 1$.

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Exercise 1.6.22 (continued)



Solution.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1-y)}}{2(1)} = \frac{2 \pm \sqrt{4y}}{2} = \frac{2 \pm 2\sqrt{y}}{2} = 1 \pm \sqrt{y}$$

where $x \geq 1$. Since $x \geq 1$, then we must have $x = 1 + \sqrt{y}$.

Interchanging x and y give $y = 1 + \sqrt{x}$, so that $y = f^{-1}(x) = 1 + \sqrt{x}$.
□

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Exercise 1.6.44

Exercise 1.6.44. Use the properties of logarithms to write the expressions as a single term: **(a)** $\ln \sec \theta + \ln \cos \theta$, **(b)** $\ln(8x + 4) - 2 \ln c$, **(c)** $3 \ln \sqrt[3]{t^2 - 1} - \ln(t + 1)$.

Solution. (a) We have

$$\begin{aligned} \ln \sec \theta + \ln \cos \theta &= \ln(\sec \theta \cos \theta) \text{ by Theorem 1.6.1(1)} \\ &= \ln\left(\frac{1}{\cos \theta} \cos \theta\right) \text{ since } \sec \theta = 1/\cos \theta \\ &= \ln 1 \text{ if } \cos \theta > 0 \\ &= \boxed{0 \text{ if } \theta \in ((2n - 1/2)\pi + (2n + 1/2)\pi) \text{ where } n \in \mathbb{Z}.} \end{aligned}$$

(b) We have

$$\begin{aligned} \ln(8x + 4) - 2 \ln c &= \ln(8x + 4) - \ln c^2 \text{ by Theorem 1.6.1(4)} \\ &= \boxed{\ln\left(\frac{8x + 4}{c^2}\right)} \text{ by Theorem 1.6.1(2).} \end{aligned}$$

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Exercise 1.6.44 (continued)

Exercise 1.6.44. Use the properties of logarithms to write the expressions as a single term: **(a)** $\ln \sec \theta + \ln \cos \theta$, **(b)** $\ln(8x + 4) - 2 \ln c$, **(c)** $3 \ln \sqrt[3]{t^2 - 1} - \ln(t + 1)$.

Solution (continued). (c) We have

$$\begin{aligned} 3 \ln \sqrt[3]{t^2 - 1} - \ln(t + 1) &= \ln\left(\sqrt[3]{t^2 - 1}\right)^3 - \ln(t + 1) \\ &\text{by Theorem 1.6.1(4)} \\ &= \ln(t^2 - 1) - \ln(t + 1) \\ &= \ln \frac{t^2 - 1}{t + 1} \text{ by Theorem 1.6.1(2)} \\ &= \ln \frac{(t - 1)(t + 1)}{t + 1} = \boxed{\ln(t - 1)}. \end{aligned}$$

□

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Exercise 1.6.54

Exercise 1.6.54. Solve for y in terms of x :

$$\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x).$$

Solution. By Theorem 1.6.1(2) we have that

$$\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x) \text{ implies } \ln\left(\frac{y^2 - 1}{y + 1}\right) = \ln(\sin x) \text{ or}$$

$$\ln\left(\frac{(y - 1)(y + 1)}{y + 1}\right) = \ln(\sin x) \text{ or } \ln(y - 1) = \ln(\sin x). \text{ We can}$$

exponentiate both sides of this equation to get $e^{\ln(y-1)} = e^{\ln(\sin x)}$ or $y - 1 = \sin x$ where $y > 1$ (we could also use the fact that the natural logarithm is one-to-one to conclude this; notice that we need $y > 1$ since the original equation involves $\ln(y - 1)$). So $y = 1 + \sin x$ where $y > 1$, or $y = 1 + \sin x$ where $\sin x > 0$. That is,

$$y = 1 + \sin x \text{ where } x \in (2n\pi, (2n + 1)\pi) \text{ for } n \in \mathbb{Z}. \quad \square$$

Example 1.6.7

Example 1.6.7. The *half-life* of a radioactive element is the time expected to pass until half of the radioactive nuclei present in a sample decay. The half-life is a constant that does not depend on the number of radioactive nuclei initially present in the sample, but only on the radioactive substance. So with the amount of radioactive nuclei present at time t given by $y = y_0 e^{-kt}$ find the half-life.

Solution. The question is $t = ?$ when $y = y_0/2$. So we consider $y_0/2 = y_0 e^{-kt}$, which implies $1/2 = e^{-kt}$. Taking a natural logarithm of both sides of the equation gives $\ln(1/2) = \ln(e^{-kt})$ or $\ln(1/2) = -kt$ or $t = (\ln(1/2))/(-k)$ or $t = (\ln(2^{-1}))/(-k) = (-\ln 2)/(-k) = \boxed{(\ln 2)/k}$.

In Example 1.5.4, we saw that for Carbon-14, $k = 1.2 \times 10^{-4}$. So the half-life of Carbon-14 is $(\ln 2)/(1.2 \times 10^{-4}) \approx 5776$ years.

Exercise 1.6.72

Exercise 1.6.72. Find the exact value of each expression: **(a)** $\cos^{-1}(1/2)$, **(b)** $\cos^{-1}(-1/\sqrt{2})$, **(c)** $\cos^{-1}(\sqrt{3}/2)$.

Solution. **(a)** With $\theta = \cos^{-1}(1/2)$, we need $\cos \theta = 1/2$ and $\theta \in [0, \pi]$. Our knowledge of special angles tells us that $\theta = \pi/3$ (see 1.3. Trigonometric Functions; see Figure 1.41). \square

(c) With $\theta = \cos^{-1}(\sqrt{3}/2)$, we need $\cos \theta = \sqrt{3}/2$ and $\theta \in [0, \pi]$. Our knowledge of special angles tells us that $\theta = \pi/6$ (see 1.3. Trigonometric Functions; see Figure 1.41). \square

Exercise 1.6.72 (continued)

Exercise 1.6.72. Find the exact value of each expression: **(a)** $\cos^{-1}(1/2)$, **(b)** $\cos^{-1}(-1/\sqrt{2})$, **(c)** $\cos^{-1}(\sqrt{3}/2)$.

Solution. **(b)** With $\theta = \cos^{-1}(-1/\sqrt{2})$, we need $\cos \theta = -1/\sqrt{2}$ and $\theta \in [0, \pi]$. Since $\cos \theta = -1/\sqrt{2} < 0$, then in fact $\theta \in [\pi/2, \pi]$. Our knowledge of special angles tells us that $\cos \pi/4 = 1/\sqrt{2}$ (see 1.3. Trigonometric Functions; see Figure 1.41), so θ must be a second quadrant angle with reference angle $\pi/4$. Hence, $\theta = 3\pi/4$. \square

