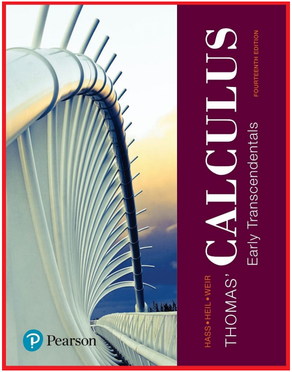
Calculus 1

Chapter 1. Functions

1.6. Inverse Functions and Logarithms—Examples and Proofs



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Exercise 1.6.10

Exercise 1.6.10. Determine from its graph if the function
$f(x) = \begin{cases} 2 - x^2, & x \leq 1 \\ x^2, & x > 1 \end{cases}$ is one-to-one.

Solution. The pieces of f are translations and reflections of the parabola $y = x^2$. The graph of $y = 2 - x^2$ is the reflection of the parabola $y = x^2$ about the x axis (to produce $y = -x^2$) which is then translated up by 2 units.

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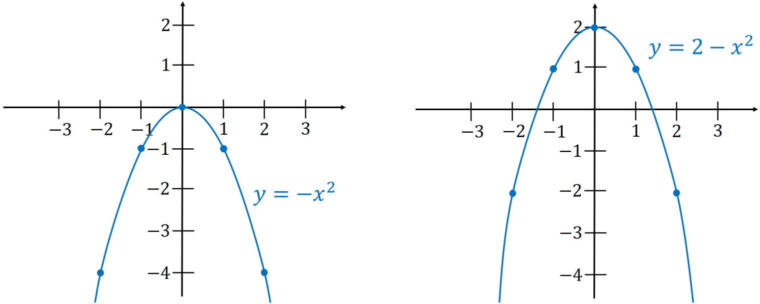

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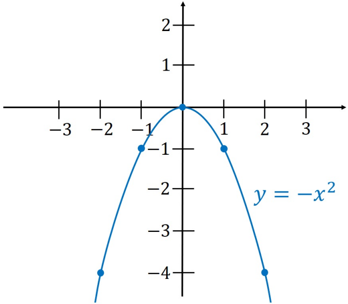
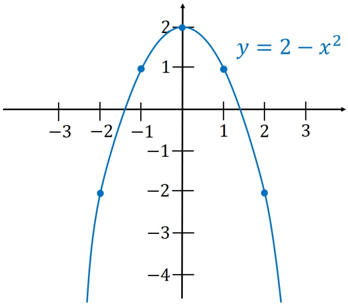
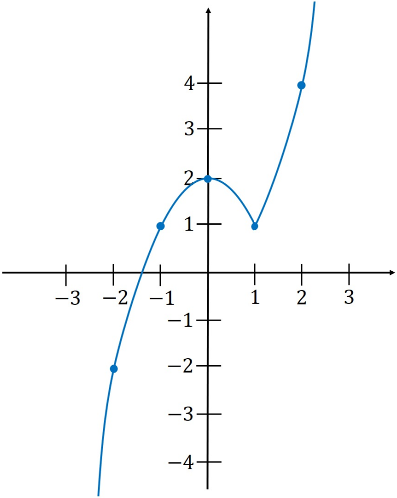




Exercise 1.6.10 (continued)

Solution (continued).
So we graph $y = 2 - x^2$ for $x \le 1$ and $y = x^2$ for $x > 1$:



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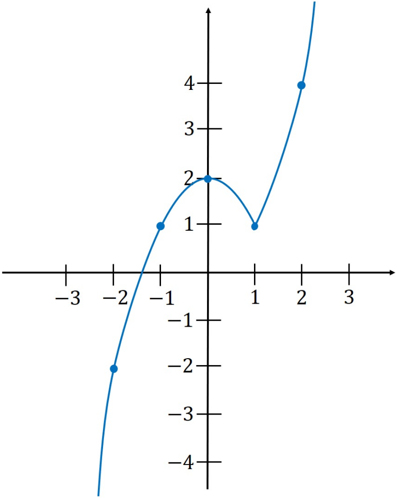

We see from the graph of $y = f(x)$ that it is $\boxed{\text{not one-to-one}}$ because, for example, the value 1 is attained at two x-values (namely, $x = -1$ and $x = 1$), and the value 2 is attained at two x-values (namely, $x = 0$ and $x = \sqrt{2}$). In addition, each value in $(1, 2)$ is attained three times! □

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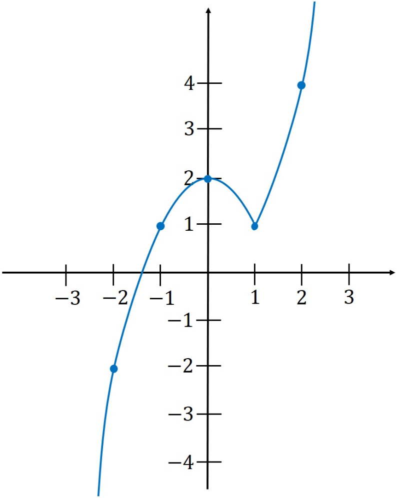

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Example 1.6.4. Find the inverse of the function $y = x^2$, $x \geq 0$. See Figure 1.59.

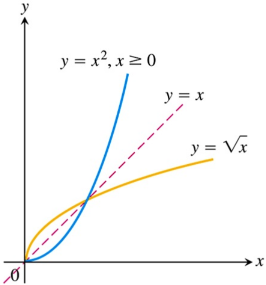

Figure 1.59

Solution. We follow the two step procedure. First, let $y = x^2$ where $x \geq 0$. Solving for x we have $\sqrt{y} = \sqrt{x^2}$ or $\sqrt{y} = |x|$ where $x \geq 0$. Since $|x| = x$ for $x \geq 0$, then $\sqrt{y} = x$.

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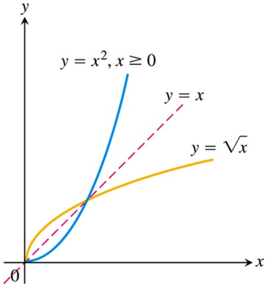

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Interchanging x and y gives $\sqrt{x} = y$, so that $\boxed{y = f^{-1}(x) = \sqrt{x}}$. $\square$

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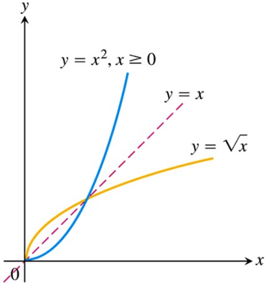

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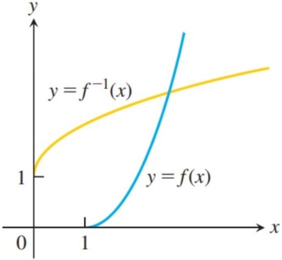
Exercise 1.6.22. Here is a graph of $f(x) = x^2 - 2x + 1$, $x \geq 1$, and its inverse. Find a formula for f^{-1}.



Solution. We follow the two step procedure. First, let $y = x^2 - 2x + 1$ where $x \geq 1$. Then $x^2 - 2x + (1 - y) = 0$ and solving for x we have by the quadratic equation that

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1-y)}}{2(1)} = \frac{2 \pm \sqrt{4y}}{2} = \frac{2 \pm 2\sqrt{y}}{2} = 1 \pm \sqrt{y}$$

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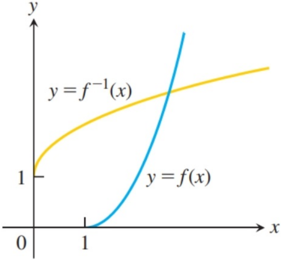

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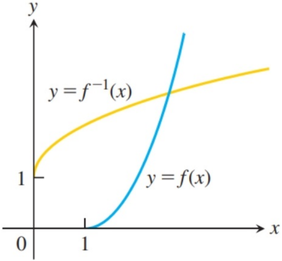

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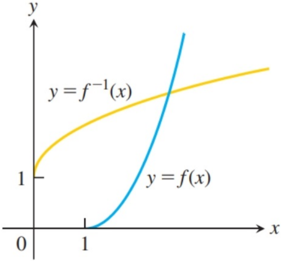

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Exercise 1.6.44. Use the properties of logarithms to write the expressions as a single term: **(a)** $\ln \sec \theta + \ln \cos \theta$, **(b)** $\ln(8x + 4) - 2 \ln c$, **(c)** $3 \ln \sqrt[3]{t^2 - 1} - \ln(t + 1)$.

Solution. **(a)** We have

$$\begin{aligned} \ln \sec \theta + \ln \cos \theta &= \ln(\sec \theta \cos \theta) \text{ by Theorem 1.6.1(1)} \\ &= \ln\left(\frac{1}{\cos \theta} \cos \theta\right) \text{ since } \sec \theta = 1/\cos \theta \\ &= \ln 1 \text{ if } \cos \theta > 0 \\ &= \boxed{0 \text{ if } \theta \in ((2n - 1/2)\pi + (2n + 1/2)\pi) \text{ where } n \in \mathbb{Z}}. \end{aligned}$$

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$$\begin{aligned} \ln(8x+4) - 2\ln c &= \ln(8x+4) - \ln c^2 \text{ by Theorem 1.6.1(4)} \\ &= \boxed{\ln\left(\frac{8x+4}{c^2}\right)} \text{ by Theorem 1.6.1(2).} \end{aligned}$$

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Solution (continued). **(c)** We have

$$\begin{aligned} 3\ln\sqrt[3]{t^2-1} - \ln(t+1) &= \ln\left(\sqrt[3]{t^2-1}\right)^3 - \ln(t+1) \\ &\quad \text{by Theorem 1.6.1(4)} \\ &= \ln(t^2-1) - \ln(t+1) \\ &= \ln\frac{t^2-1}{t+1} \text{ by Theorem 1.6.1(2)} \\ &= \ln\frac{(t-1)(t+1)}{t+1} = \boxed{\ln(t-1)}. \end{aligned}$$

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Exercise 1.6.54. Solve for y in terms of x:
$\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x)$.

Solution. By Theorem 1.6.1(2) we have that
$\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x)$ implies $\ln\left(\frac{y^2 - 1}{y + 1}\right) = \ln(\sin x)$ or
$\ln\left(\frac{(y - 1)(y + 1)}{y + 1}\right) = \ln(\sin x)$ or $\ln(y - 1) = \ln(\sin x)$.

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Example 1.6.7

Example 1.6.7. The *half-life* of a radioactive element is the time expected to pass until half of the radioactive nuclei present in a sample decay. The half-life is a constant that does not depend on the number of radioactive nuclei initially present in the sample, but only on the radioactive substance. So with the amount of radioactive nuclei present at time t given by $y = y_0 e^{-kt}$ find the half-life.

Solution. The question is $t = ?$ when $y = y_0/2$. So we consider $y_0/2 = y_0 e^{-kt}$, which implies $1/2 = e^{-kt}$. Taking a natural logarithm of both sides of the equation gives $\ln(1/2) = \ln(e^{-kt})$ or $\ln(1/2) = -kt$ or $t = (\ln(1/2))/(-k)$ or $t = (\ln(2^{-1}))/(-k) = (-\ln 2)/(-k) = \boxed{(\ln 2)/k}$.

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Exercise 1.6.72. Find the exact value of each expression: **(a)** $\cos^{-1}(1/2)$, **(b)** $\cos^{-1}(-1/\sqrt{2})$, **(c)** $\cos^{-1}(\sqrt{3}/2)$.

Solution. **(a)** With $\theta = \cos^{-1}(1/2)$, we need $\cos\theta = 1/2$ and $\theta \in [0, \pi]$. Our knowledge of special angles tells us that $\boxed{\theta = \pi/3}$ (see 1.3. Trigonometric Functions; see Figure 1.41). □

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(c) With $\theta = \cos^{-1}(\sqrt{3}/2)$, we need $\cos\theta = \sqrt{3}/2$ and $\theta \in [0, \pi]$. Our knowledge of special angles tells us that $\boxed{\theta = \pi/6}$ (see 1.3. Trigonometric Functions; see Figure 1.41). □

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Exercise 1.6.72. Find the exact value of each expression: **(a)** $\cos^{-1}(1/2)$, **(b)** $\cos^{-1}(-1/\sqrt{2})$, **(c)** $\cos^{-1}(\sqrt{3}/2)$.

Solution. **(b)** With $\theta = \cos^{-1}(-1/\sqrt{2})$, we need $\cos\theta = -1/\sqrt{2}$ and $\theta \in [0, \pi]$. Since $\cos\theta = -1/\sqrt{2} < 0$, then in fact $\theta \in [\pi/2, \pi]$. Our knowledge of special angles tells us that $\cos\pi/4 = 1/\sqrt{2}$ (see 1.3. Trigonometric Functions; see Figure 1.41), so θ must be a second quadrant angle with reference angle $\pi/4$. Hence, $\boxed{\theta = 3\pi/4}$. □