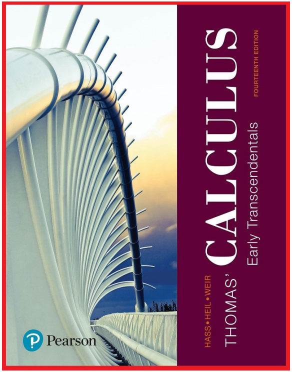
Calculus 1

Chapter 2. Limits and Continuity

2.1. Rates of Change and Tangent Lines to Curves—Examples and Proofs



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(b) After 1 second, the rock has fallen $y = 16(1)^2 = 16$ feet, so between second 1 and second 2 the average speed is $\Delta y/\Delta t = ((64 - 16) \text{ ft})/((2 - 1) \text{ sec}) = \boxed{48 \text{ ft/sec}}$.

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$\Delta y/\Delta t = ((256 - 64) \text{ ft})/((4 - 2) \text{ sec}) = \boxed{96 \text{ ft/sec}}$. □

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Exercise 2.1.4.

Exercise 2.1.4. Find the average rate of change of the function $g(t) = 2 + \cos t$ over the given interval: **(a)** $[0, \pi]$, **(b)** $[-\pi, \pi]$.

Solution. By definition, the average rate of change of $y = g(t)$ with respect to t over the interval $[t_1, t_2]$ is

$$\frac{\Delta y}{\Delta t} = \frac{g(t_2) - g(t_1)}{t_2 - t_1} = \frac{g(t_1 + h) - g(t_1)}{h}$$

where $h = t_2 - t_1 \neq 0$.

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(a) So with $g(t) = 2 + \cos t$, $t_1 = 0$, $t_2 = \pi$, and $h = t_2 - t_1 = \pi - 0 = \pi$ we have the average rate of change of g on $[0, \pi]$ is

$$\frac{\Delta y}{\Delta t} = \frac{g(t_2) - g(t_1)}{t_2 - t_1} = \frac{g(t_1 + h) - g(t_1)}{h} = \frac{g(\pi) - g(0)}{\pi - 0}$$

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Exercise 2.1.4 (continued).

Exercise 2.1.4. Find the average rate of change of the function $g(t) = 2 + \cos t$ over the given interval: **(a)** $[0, \pi]$, **(b)** $[-\pi, \pi]$.

Solution (continued). (b) With $g(t) = 2 + \cos t$, $t_1 = -\pi$, $t_2 = \pi$, and $h = t_2 - t_1 = \pi - (-\pi) = 2\pi$ we have the average rate of change of g on $[-\pi, \pi]$ is

$$\frac{\Delta y}{\Delta t} = \frac{g(t_2) - g(t_1)}{t_2 - t_1} = \frac{g(t_1 + h) - g(t_1)}{h} = \frac{g(\pi) - g(-\pi)}{\pi - (-\pi)}$$

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□

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Example 2.1.3. Find the slope of the parabola $y = x^2$ at the point $P(2,4)$. Write an equation for the tangent to the parabola at this point. **Hint:** Choose a second point $Q(2+h,(2+h)^2)$ on the curve (where $h \neq 0$) and compute the slope of the secant line PQ. *Guess* what happens to the slope of the secant line when h is close to 0.

Solution. With the points $P(2,4)$ and $Q(2+h,(2+h)^2)$, the slope of the secant line to the parabola through points P and Q is (since $h \neq 0$):

$$\begin{aligned} m &= \frac{(2+h)^2-(4)}{(2+h)-(2)} \\ &= \frac{4+4h+h^2-4}{h} \\ &= \frac{4h+h^2}{h} \\ &= \frac{h(4+h)}{h} \\ &= 4+h. \end{aligned}$$

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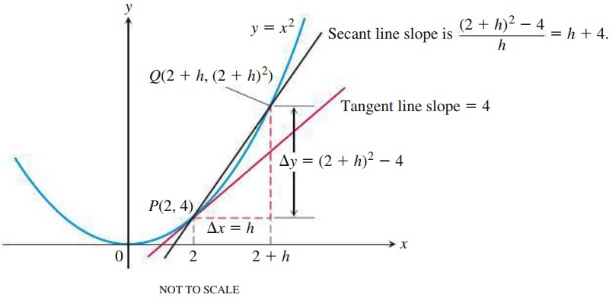

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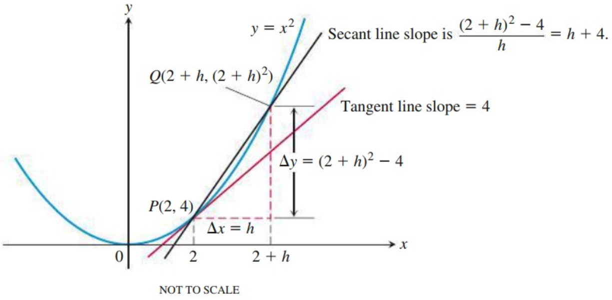

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Example 2.1.3. Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

Solution (continued). Since the slope of the secant line through the points $P(2, 4)$ and $Q(2 + h, (2 + h)^2)$ is $m = 4 + h$ where $h \neq 0$, then for h "close to" 0 we have point P "close to" point Q and the secant line (which has a slope "close to" $4 + (0) = 4$) is "close to" the line tangent to the parabola at point P.

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The tangent line to the parabola at point $P(2,4) = (x_1, y_1)$, by the point-slope formula for a line $y - y_1 = m(x - x_1)$, is then $y - (4) = 4(x-2)$ or $y = 4x - 8 + 4$ or $\boxed{y = 4x - 4}$. □

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Exercise 2.1.14

Exercise 2.1.14. Use the method of Example 2.1.3 to find the slope of the curve $y = x^3 - 3x^2 + 4$ at the point $P(2, 0)$. Write an equation for the tangent line to the curve at point P.

Solution. As in Example 2.1.3, we choose a second point on the graph of the curve, $Q(2 + h, (2 + h)^3 - 3(2 + h)^2 + 4)$. The slope of the secant line to the curve through points P and Q is (since $h \neq 0$):

$$m = \frac{((2+h)^3 - 3(2+h)^2 + 4) - (0)}{(2+h) - (2)} =$$

$$\frac{(8 + 12h + 6h^2 + h^3) - 3(4 + 4h + h^2) + 4}{h} = \frac{h^3 + 3h^2}{h} = \frac{h^2(h+3)}{h} =$$

$$h(h+3) = h^2 + 3h.$$

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$h(h+3) = h^2 + 3h$. Then for h "close to" 0 we have point P "close to" point Q and the secant line (which has a slope "close to" $(0)^2 + 3(0) = 0$) is "close to" the line tangent to the parabola at point P. So we *guess* that the slope of the tangent line, and hence the

slope of the curve at point $P(2,0)$ is $m = 0$.

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Solution (continued). The tangent line to the curve at point $P(2, 0) = (x_1, y_1)$, by the point-slope formula for a line $y - y_1 = m(x - x_1)$, is then $y - (0) = 0(x - 2)$ or $\boxed{y = 0}$. □

Exercise 2.1.14 (continued)

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Exercise 2.1.22

Exercise 2.1.22. Make a table of values for the function $F(x) = (x+2)/(x-2)$ at the points $x = 1.2$, $x = 101/100$, $x = 1001/1000$, $x = 10001/10000$, and $x = 1$. **(a)** Find the average rate of change of $F(x)$ over the intervals $[1, x]$ for each $x \neq 1$ in your table. **(b)** Extending the table if necessary, try to determine the rate of change of $F(x)$ at $x = 1$ (that is, guess).

Solution. We substitute into the function and then approximate to six decimal places.

x	$F(x)$
1.2	$\frac{(1.2)+2}{(1.2)-2} = \frac{3.2}{-.8} = -4$
$\frac{101}{100} = 1.01$	$\frac{(101/100)+2}{(101/100)-2} = \frac{(1.01)+2}{(1.01)-2} = \frac{3.01}{-0.99} \approx -3.040404$
$\frac{1001}{1000} = 1.001$	$\frac{(1001/1000)+2}{(1001/1000)-2} = \frac{(1.001)+2}{(1.001)-2} = \frac{3.001}{-0.999} \approx -3.004004$
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1	$\frac{(1)+2}{(1)-2} = -3$

Exercise 2.1.22 (continued)

Solution (continued). (a) The *average rate of change* of $y = F(x)$ with respect to x over the interval $[x_1, x_2]$ is $\frac{\Delta y}{\Delta x} = \frac{F(x_2) - F(x_1)}{x_2 - x_1} = \frac{F(x_1 + h) - F(x_1)}{h}$ where $h = x_2 - x_1 \neq 0$. So we have the average rates of change:

Interval $[1, x]$	$\frac{\Delta y}{\Delta x} = \frac{F(x)-F(1)}{x-1}$
$[1, 1.2]$	$\frac{F(1.2)-F(1)}{(1.2)-1} = \frac{(-4)-(-3)}{(1.2)-(1)} = \frac{-1}{0.2} = -5$
$[1, 1.01]$	$\frac{F(1.01)-F(1)}{(1.01)-1} \approx \frac{(-3.040404)-(-3)}{(1.01)-(1)} = \frac{-0.040404}{0.01} = -4.0404$
$[1, 1.001]$	$\frac{F(1.001)-F(1)}{(1.001)-1} \approx \frac{(-3.004004)-(-3)}{(1.001)-(1)} = \frac{-0.004004}{0.001} = -4.004$
$[1, 1.0001]$	$\frac{F(1.0001)-F(1)}{(1.0001)-1} \approx \frac{(3.000400)-(-3)}{(1.0001)-(1)} = \frac{-0.000400}{0.0001} = -4.000$

(b) We see the average rate of change tending to -4 on the interval $[1, x] = [1, 1 + h]$ where $h = 0.2, 0.1, 0.001$, and 0.0001, so we guess that the instantaneous rate of change of F at $x = 1$ is $\boxed{-4}$. $\square$

Exercise 2.1.22 (continued)

Solution (continued). (a) The *average rate of change* of $y = F(x)$ with respect to x over the interval $[x_1, x_2]$ is
$\frac{\Delta y}{\Delta x} = \frac{F(x_2) - F(x_1)}{x_2 - x_1} = \frac{F(x_1 + h) - F(x_1)}{h}$ where $h = x_2 - x_1 \neq 0$. So we have the average rates of change:

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$[1, 1.01]$	$\frac{F(1.01)-F(1)}{(1.01)-1} \approx \frac{(-3.040404)-(-3)}{(1.01)-(1)} = \frac{-0.040404}{0.01} = -4.0404$
$[1, 1.001]$	$\frac{F(1.001)-F(1)}{(1.001)-1} \approx \frac{(-3.004004)-(-3)}{(1.001)-(1)} = \frac{-0.004004}{0.001} = -4.004$
$[1, 1.0001]$	$\frac{F(1.0001)-F(1)}{(1.0001)-1} \approx \frac{(3.000400)-(-3)}{(1.0001)-(1)} = \frac{-0.000400}{0.0001} = -4.000$

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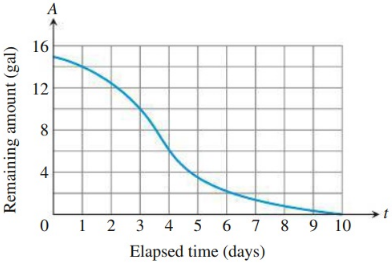
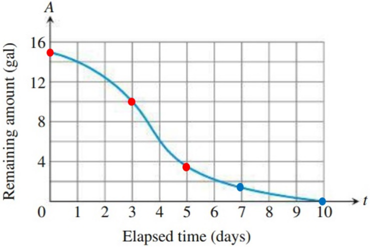
Exercise 2.1.26

Exercise 2.1.26. The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for t days. **(a)** Estimate the average rate of gasoline consumption over the time intervals $[0, 3]$, $[0, 5]$, and $[7, 10]$. **(b)** Estimate the instantaneous rate of gasoline consumption at the times $t = 1$, $t = 4$, and $t = 8$. **(c)** Estimate the maximum rate of gasoline consumption and the specific time at which it occurs. Notice that the rate of gasoline *consumption* per day is the negative of the rate of change of the amount of gas *remaining* per day.



Exercise 2.1.26 (continued 1)

Solution. **(a)** We estimate the value of the graphed function at $t = 0$ days is $A = 15$ gal, at $t = 3$ days is $A = 10$ gal, at $t = 5$ days is 3.5 gal, at $t = 7$ days is $A = 1.5$ gal, and at $t = 10$ days is $A = 0$ gal.



On the time interval $[0, 3]$ the average rate of gasoline consumption is $-\frac{\Delta A}{\Delta t} = -\frac{(10)-(15)}{(3)-(0)} = \frac{5}{3} \approx \boxed{1.67 \text{ gal/day}}$. On the interval $[0, 5]$ the average rate is $-\frac{\Delta A}{\Delta t} = -\frac{(3.5)-(15)}{(5)-(0)} = \frac{11.5}{5} = \boxed{2.3 \text{ gal/day}}$. On the interval $[7, 10]$ the average rate is $-\frac{\Delta A}{\Delta t} = -\frac{(0)-(1.5)}{(10)-(7)} = -\frac{-1.5}{3} = \boxed{0.5 \text{ gal/day}}$. $\square$

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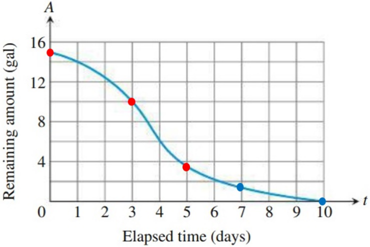

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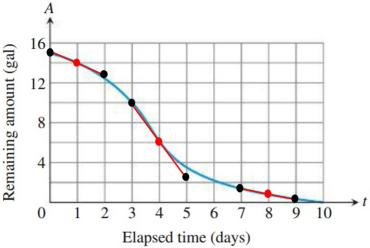
Exercise 2.1.26 (continued 2)

Solution (continued). (b) To estimate the instantaneous rate of gasoline consumption, we estimate the negative of the slope of the line tangent to the graph at the times $t = 1$, $t = 4$, and $t = 8$ (this is subjective):



At $t = 1$, we estimate the negative of the slope as $-\Delta A/\Delta t = -(13 - 15)/(2 - 0) = \boxed{1 \text{ gal/day}}$. At $t = 4$, we estimate the negative of the slope as $-\Delta A/\Delta t = -(2.5 - 10)/(5 - 3) = \boxed{15/4 \text{ gal/day}}$. At $t = 8$, we estimate the negative of the slope as $-\Delta A/\Delta t = -(0.3 - 1.5)/(9 - 7) = \boxed{3/5 \text{ gal/day}}$. $\square$

Exercise 2.1.26 (continued 2)

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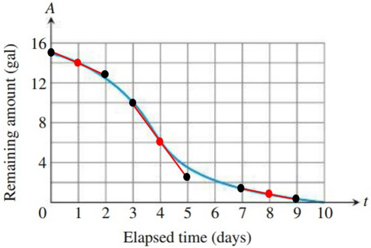

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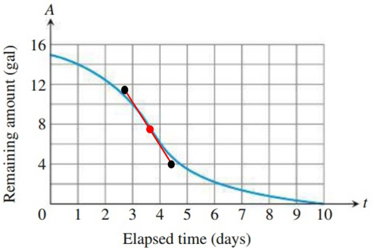
Exercise 2.1.26 (continued 3)

Solution (continued). (c) To estimate the maximum rate of gasoline consumption and the specific time at which it occurs, we look for the point of steepest decrease of the graph (this is subjective too). I estimate that this occurs at $t = 3.7$ days and that the rate of gasoline consumption is $-\Delta A/\Delta t = -(4 - 11.5)/(4.5 - 2.8) \approx \boxed{4.41 \text{ gal/day}}$.



□