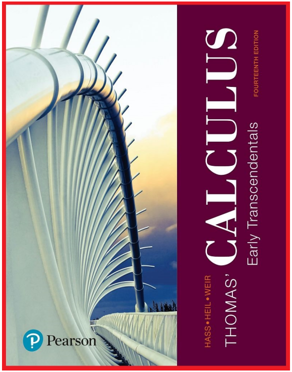
Calculus 1

Chapter 2. Limits and Continuity

2.2. Limit of a Function and Limit Laws—Examples and Proofs



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Example 2.2.A. Use the above technique to evaluate $\lim_{x \to 3} \frac{x^2 - 4x + 3}{x - 3}$.

Solution. Let $f(x) = \frac{x^2 - 4x + 3}{x - 3} = \frac{(x-1)(x-3)}{x-3}$. Then $f(x) = x - 1$ if $x \neq 3$, and so $\lim_{x \to 3} \frac{x^2 - 4x + 3}{x - 3} = \lim_{x \to 3} x - 1$ if $x \neq 3$.

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For x "close to" 3, $x - 1$ is close to $(3) - 1 = 2$. So $\lim_{x \to 3} \frac{x^2 - 4x + 3}{x - 3} = \lim_{x \to 3} x - 1 = (3) - 1 = \boxed{2}$. □

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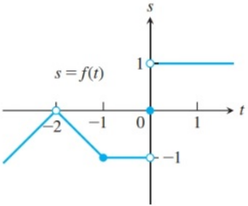
Exercise 2.2.2.

Exercise 2.2.2. For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.

(a) $\lim_{t \to -2} f(t)$,
(b) $\lim_{t \to -1} f(t)$,
(c) $\lim_{t \to 0} f(t)$,
(d) $\lim_{t \to -0.5} f(t)$



Solution. **(a)** When considering $\lim_{t \to -2} f(t)$, we apply Dr. Bobs Anthropomorphic Definition of Limit and see if there is a point that the graph of $s = f(t)$ tries to pass through for t near -2.

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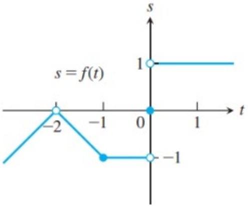

Solution. **(a)** When considering $\lim_{t\to -2} f(t)$, we apply Dr. Bobs Anthropomorphic Definition of Limit and see if there is a point that the graph of $s = f(t)$ tries to pass through for t near -2. There is such a point; the graph of $s = f(t)$ tries to pass through the point $(-2, 0)$ (and it fails). So we conclude that $\boxed{\lim_{t\to -2} f(t) = 0}$. □

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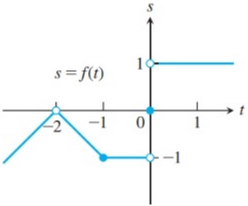

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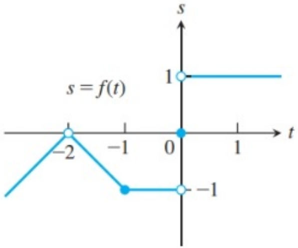
Exercise 2.2.2 (continued 1).



Solution (continued). (b) When considering $\lim_{t \to -1} f(t)$, we apply Dr. Bobs Anthropomorphic Definition of Limit and see if there is a point that the graph of $s = f(t)$ tries to pass through for t near -1. There is such a point; the graph of $s = f(t)$ tries to pass through the point $(-1, -1)$ (and it succeeds). So we conclude that $\boxed{\lim_{t \to -1} f(t) = -1}$. □

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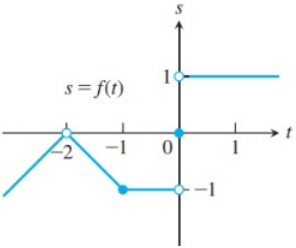
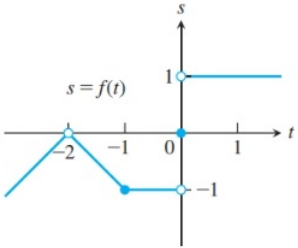


Solution (continued). (b) When considering $\lim_{t \to -1} f(t)$, we apply Dr. Bobs Anthropomorphic Definition of Limit and see if there is a point that the graph of $s = f(t)$ tries to pass through for t near -1. There is such a point; the graph of $s = f(t)$ tries to pass through the point $(-1, -1)$ (and it succeeds). So we conclude that $\boxed{\lim_{t \to -1} f(t) = -1}$. □

Exercise 2.2.2 (continued 2).



Solution (continued). (c) When considering $\lim_{t \to 0} f(t)$, we apply Dr. Bobs Anthropomorphic Definition of Limit and see if there is a point that the graph of $s = f(t)$ tries to pass through for t near 0. There is no such point! The graph approaches the point $(0, -1)$ when t is close to 0 and less than 0, but the graph approaches the point $(0, 1)$ when t is close to 0 and greater than 0 (it doesn't matter what happens *at* 0). Since there is no (single) such point, we conclude that $\boxed{\lim_{t \to 0} f(t) \text{ does not exist}}$. □

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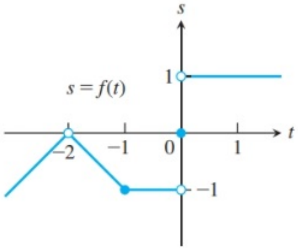

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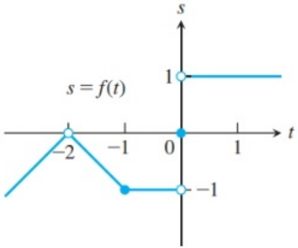
Exercise 2.2.2 (continued 3).



Solution (continued). (d) When considering $\lim_{t \to -0.5} f(t)$, we apply Dr. Bobs Anthropomorphic Definition of Limit and see if there is a point that the graph of $s = f(t)$ tries to pass through for t near -0.5. There is such a point; the graph of $s = f(t)$ tries to pass through the point $(-0.5, -1)$ (and it succeeds). So we conclude that $\boxed{\lim_{t \to -0.5} f(t) = -1}$. $\square$

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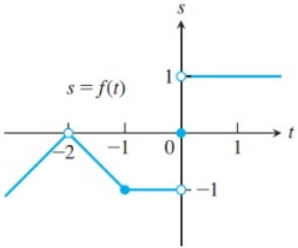

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Exercise 2.2.52

Exercise 2.2.52. Let $\lim_{x\to 1} h(x) = 5$, $\lim_{x\to 1} p(x) = 1$, and $\lim_{x\to 1} r(x) = 2$. Name the rules in Theorem 2.1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\begin{aligned}
\lim_{x\to 1} \frac{\sqrt{5h(x)}}{p(x)(4 - r(x))} &= \frac{\lim_{x\to 1} \sqrt{5h(x)}}{\lim_{x\to 1}(p(x)(4 - r(x)))} && (a) \\
&= \frac{\sqrt{\lim_{x\to 1} 5h(x)}}{(\lim_{x\to 1} p(x))\,(\lim_{x\to 1}(4 - r(x)))} && (b) \\
&= \frac{\sqrt{5 \lim_{x\to 1} h(x)}}{(\lim_{x\to 1} p(x))\,(\lim_{x\to 1} 4 - \lim_{x\to 1} r(x))} && (c) \\
&= \frac{\sqrt{(5)(5)}}{(1)(4 - 2)} = \frac{5}{2}.
\end{aligned}$$

Solution. In (a) we have used the fact that the limit of a quotient is the quotient of the limits (Theorem 2.1(5), Quotient Rule), provided the denominator doesn't have a limit of 0.

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Exercise 2.2.52 (continued)

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\end{aligned}$$

Solution (continued). In (c) we, we have used the Constant Multiple Rule (Theorem 2.1(3)) in the numerator, and the Difference Rule (Theorem 2.1(2)) in the denominator. In the last step we have substituted in the given limit values (and used Note 2.2.A). □

Theorem 2.2

Theorem 2.2. Limits of Polynomials Can Be Found by Substitution.
If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0.$$

Proof. We apply Theorem 2.1, "Limit Rules." We have

$$\lim_{x \to c} P(x) = \lim_{x \to c} (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)$$

by the definition of P

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Proof. We apply Theorem 2.1, "Limit Rules." We have

$$\begin{aligned}\lim_{x\to c} P(x) &= \lim_{x\to c}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) \\ &\quad \text{by the definition of } P \\ &= \lim_{x\to c} a_n x^n + \lim_{x\to c} a_{n-1}x^{n-1} + \cdots + \lim_{x\to c} a_1 x + \lim_{x\to c} a_0 \\ &\quad \text{by repeated use of the Sum Rule, Theorem 2.1(1)}\end{aligned}$$

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&\quad \text{by the definition of } P \\
&= \lim_{x\to c} a_n x^n + \lim_{x\to c} a_{n-1}x^{n-1} + \cdots + \lim_{x\to c} a_1 x + \lim_{x\to c} a_0 \\
&\quad \text{by repeated use of the Sum Rule, Theorem 2.1(1)} \\
&= a_n \lim_{x\to c} x^n + a_{n-1}\lim_{x\to c} x^{n-1} + \cdots + a_1 \lim_{x\to c} x + \lim_{x\to c} a_0 \\
&\quad \text{by repeated use of the Constant Multiple Rule,} \\
&\quad \text{Theorem 2.1(3)} \ldots
\end{aligned}$$

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&\quad \text{by the definition of } P \\
&= \lim_{x\to c} a_n x^n + \lim_{x\to c} a_{n-1}x^{n-1} + \cdots + \lim_{x\to c} a_1 x + \lim_{x\to c} a_0 \\
&\quad \text{by repeated use of the Sum Rule, Theorem 2.1(1)} \\
&= a_n \lim_{x\to c} x^n + a_{n-1} \lim_{x\to c} x^{n-1} + \cdots + a_1 \lim_{x\to c} x + \lim_{x\to c} a_0 \\
&\quad \text{by repeated use of the Constant Multiple Rule,} \\
&\quad \text{Theorem 2.1(3)}\ldots
\end{aligned}$$

Theorem 2.2 (continued)

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If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0.$$

Proof (continued).

$$\begin{aligned}\lim_{x \to c} P(x) &= a_n \lim_{x \to c} x^n + a_{n-1} \lim_{x \to c} x^{n-1} + \cdots + a_1 \lim_{x \to c} x + \lim_{x \to c} a_0 \\ &= a_n \left(\lim_{x \to c} x\right)^n + a_{n-1} \left(\lim_{x \to c} x\right)^{n-1} + \cdots + a_1 \left(\lim_{x \to c} x\right) + \lim_{x \to c} a_0\end{aligned}$$

by repeated use of the Power Rule, Theorem 2.1(6)

Theorem 2.2 (continued)

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If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0.$$

Proof (continued).

$$\begin{aligned}
\lim_{x\to c} P(x) &= a_n \lim_{x\to c} x^n + a_{n-1} \lim_{x\to c} x^{n-1} + \cdots + a_1 \lim_{x\to c} x + \lim_{x\to c} a_0 \\
&= a_n \left(\lim_{x\to c} x\right)^n + a_{n-1} \left(\lim_{x\to c} x\right)^{n-1} + \cdots + a_1 \left(\lim_{x\to c} x\right) + \lim_{x\to c} a_0 \\
&\quad \text{by repeated use of the Power Rule, Theorem 2.1(6)} \\
&= a_n(c)^n + a_{n-1}(c)^{n-1} + \cdots + a_1(c) + a_0 \text{ by Note 2.2.A} \\
&= a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0,
\end{aligned}$$

as claimed. □

Theorem 2.2 (continued)

Theorem 2.2. Limits of Polynomials Can Be Found by Substitution. If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0.$$

Proof (continued).

$$\begin{aligned}
\lim_{x\to c} P(x) &= a_n \lim_{x\to c} x^n + a_{n-1}\lim_{x\to c} x^{n-1} + \cdots + a_1 \lim_{x\to c} x + \lim_{x\to c} a_0 \\
&= a_n \left(\lim_{x\to c} x\right)^n + a_{n-1}\left(\lim_{x\to c} x\right)^{n-1} + \cdots + a_1 \left(\lim_{x\to c} x\right) + \lim_{x\to c} a_0 \\
&\quad \text{by repeated use of the Power Rule, Theorem 2.1(6)} \\
&= a_n(c)^n + a_{n-1}(c)^{n-1} + \cdots + a_1(c) + a_0 \text{ by Note 2.2.A} \\
&= a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0,
\end{aligned}$$

as claimed. □

Theorem 2.3

Theorem 2.3. Limits of Rational Functions Can Be Found by Substituting IF the Limit of the Denominator Is Not Zero.
If P and Q are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{\lim_{x \to c} P(x)}{\lim_{x \to c} Q(x)} = \frac{P(c)}{Q(c)}.$$

Proof. We apply Theorem 2.1 and Theorem 2.2. Since P and Q are polynomials then, by Theorem 2.2, $\lim_{x \to c} P(x) = P(c)$ and $\lim_{x \to c} Q(x) = Q(c)$. By hypothesis $Q(c) \neq 0$, so by the Quotient Rule (Theorem 2.1(5)) we have

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{\lim_{x \to c} P(x)}{\lim_{x \to c} Q(x)} = \frac{P(c)}{Q(c)},$$

as claimed. □

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Theorem 2.3. Limits of Rational Functions Can Be Found by Substituting IF the Limit of the Denominator Is Not Zero.
If P and Q are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{\lim_{x \to c} P(x)}{\lim_{x \to c} Q(x)} = \frac{P(c)}{Q(c)}.$$

Proof. We apply Theorem 2.1 and Theorem 2.2. Since P and Q are polynomials then, by Theorem 2.2, $\lim_{x \to c} P(x) = P(c)$ and $\lim_{x \to c} Q(x) = Q(c)$. By hypothesis $Q(c) \neq 0$, so by the Quotient Rule (Theorem 2.1(5)) we have

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{\lim_{x \to c} P(x)}{\lim_{x \to c} Q(x)} = \frac{P(c)}{Q(c)},$$

as claimed. □

Exercise 2.2.14

Exercise 2.2.14. Evaluate $\lim_{x \to -2} x^3 - 2x^2 + 4x + 8$.

Solution. Since $P(x) = x^3 - 2x^2 + 4x + 8$ is a polynomial function, then by Theorem 2.2

$$\lim_{x \to -2} P(x) = \lim_{x \to -2} x^3 - 2x^2 + 4x + 8 = P(-2)$$

$$= (-2)^3 - 2(-2)^2 + 4(-2) + 8 = -8 - 8 - 8 + 8 = -16.$$

□

Exercise 2.2.14

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□

Exercise 2.2.18

Exercise 2.2.18. Evaluate $\lim_{y\to 2} \dfrac{y+2}{y^2+5y+6}$.

Solution. Since $R(y) = \dfrac{P(y)}{Q(y)} = \dfrac{y+2}{y^2+5y+6}$ is a rational function (with the numerator as the polynomial function $P(y) = y+2$ and the denominator as the polynomial function $Q(y) = y^2+5y+6$) where, by Theorem 2.2, $\lim_{y\to 2} P(y) = \lim_{y\to 2} y+2 = P(2) = (2)+2 = 4$ and

$\lim_{y\to 2} Q(y) = \lim_{y\to 2} y^2+5y+6 = Q(2) = (2)^2+5(2)+6 = 4+10+6 = 20 \neq 0,$

then by Theorem 2.3

$$\lim_{y\to 2} R(y) = \lim_{y\to 2} \frac{P(y)}{Q(y)} = \frac{\lim_{y\to 2} P(y)}{\lim_{y\to 2} Q(y)} = \frac{P(2)}{Q(2)} = \frac{4}{20} = \boxed{\frac{1}{5}}.$$

□

Exercise 2.2.18

Exercise 2.2.18. Evaluate $\lim_{y\to 2} \dfrac{y+2}{y^2+5y+6}$.

Solution. Since $R(y) = \dfrac{P(y)}{Q(y)} = \dfrac{y+2}{y^2+5y+6}$ is a rational function (with the numerator as the polynomial function $P(y) = y+2$ and the denominator as the polynomial function $Q(y) = y^2+5y+6$) where, by Theorem 2.2, $\lim_{y\to 2} P(y) = \lim_{y\to 2} y+2 = P(2) = (2)+2 = 4$ and

$$\lim_{y\to 2} Q(y) = \lim_{y\to 2} y^2+5y+6 = Q(2) = (2)^2+5(2)+6 = 4+10+6 = 20 \neq 0,$$

then by Theorem 2.3

$$\lim_{y\to 2} R(y) = \lim_{y\to 2} \frac{P(y)}{Q(y)} = \frac{\lim_{y\to 2} P(y)}{\lim_{y\to 2} Q(y)} = \frac{P(2)}{Q(2)} = \frac{4}{20} = \boxed{\frac{1}{5}}.$$

□

Exercise 2.2.34

Exercise 2.2.34. Evaluate $\lim_{v \to 2} \dfrac{v^3 - 8}{v^4 - 16}$.

Solution. We apply Dr. Bob's Limit Theorem (Theorem 2.2.A), which allows us to **F**actor/**C**ancel/**S**ubstitute. Recall that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ (the difference of cubes identity), so we have that

$v^3 - 8 = v^3 - 2^3 = (v-2)(v^2 + 2v + 2^2) = (v-2)(v^2 + 2v + 4)$.

Exercise 2.2.34

Exercise 2.2.34. Evaluate $\lim_{v \to 2} \frac{v^3 - 8}{v^4 - 16}$.

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$$\lim_{v \to 2} \frac{v^3 - 8}{v^4 - 16} = \lim_{v \to 2} \frac{(v - 2)(v^2 + 2v + 4)}{(v^2 - 4)(v^2 + 4)}$$ using the difference of cubes identity and the fact that $v^4 - 16 = (v^2)^2 - 4^2$ is a difference of two squares

Exercise 2.2.34

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$$\lim_{v\to 2}\frac{v^3-8}{v^4-16} = \lim_{v\to 2}\frac{(v-2)(v^2+2v+4)}{(v^2-4)(v^2+4)}$$ using the difference of cubes identity and the fact that $v^4-16=(v^2)^2-4^2$ is a difference of two squares

$$= \lim_{v\to 2}\frac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)}$$ since $v^2-4=v^2-2^2$ is a difference of two squares $(*)$

Exercise 2.2.34

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$$\begin{aligned}
\lim_{v\to 2}\frac{v^3-8}{v^4-16} &= \lim_{v\to 2}\frac{(v-2)(v^2+2v+4)}{(v^2-4)(v^2+4)} \text{ using the difference of cubes identity and the fact that } v^4-16=(v^2)^2-4^2 \text{ is a difference of two squares} \\
&= \lim_{v\to 2}\frac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)} \text{ since } v^2-4=v^2-2^2 \text{ is a difference of two squares } (*)
\end{aligned}$$

Exercise 2.2.34 (continued 1)

Exercise 2.2.34. Evaluate $\lim_{v\to 2} \frac{v^3-8}{v^4-16}$.

Solution (continued).

$$\begin{aligned}
\lim_{v\to 2} \frac{v^3-8}{v^4-16} &= \lim_{v\to 2} \frac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)} \\
&= \lim_{v\to 2} \frac{(v^2+2v+4)}{(v+2)(v^2+4)} \text{ cancelling the factors } (v-2) \\
&\quad \text{and applying Dr. Bob's Limit Theorem } (**) \\
&= \frac{(2)^2+2(2)+4}{((2)+2)((2)^2+4)} \text{ substituting, by Theorem 2.3} \\
&\quad \text{(Limits of Rational Functions) } (***) \\
&= \frac{4+4+4}{(4)(4+4)} = \frac{12}{32} = \boxed{\frac{3}{8}}. \qquad \square
\end{aligned}$$

Exercise 2.2.34 (continued 1)

Exercise 2.2.34. Evaluate $\lim_{v\to 2} \dfrac{v^3-8}{v^4-16}$.

Solution (continued).

$$\begin{aligned}
\lim_{v\to 2} \frac{v^3-8}{v^4-16} &= \lim_{v\to 2} \frac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)} \\
&= \lim_{v\to 2} \frac{(v^2+2v+4)}{(v+2)(v^2+4)} \text{ cancelling the factors } (v-2) \\
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&= \frac{4+4+4}{(4)(4+4)} = \frac{12}{32} = \boxed{\frac{3}{8}}. \qquad \square
\end{aligned}$$

Exercise 2.2.34 (continued 2)

Note. We repeat the starred equations here to emphasize that we have Factored (at step $(*)$), Cancelled (at step $(**)$), and Substituted (at step $(***)$):

$$\begin{aligned}\lim_{v\to 2}\frac{v^3-8}{v^4-16} &= \lim_{v\to 2}\frac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)} \quad (*)\\ &= \lim_{v\to 2}\frac{(v^2+2v+4)}{(v+2)(v^2+4)} \quad (**)\\ &= \frac{(2)^2+2(2)+4}{((2)+2)((2)^2+4)} = 3/8. \quad (***)\end{aligned}$$

This is a common way to evaluate limits! Here, the steps are justified by algebra (the Factoring), Dr. Bob's Limit Theorem (the Cancellation), and Theorem 2.3 (the Substitution). One must be careful when substituting to avoid division by zero, square roots of negatives, logarithms of zero or negatives, or other forbidden mathematical maneuvers! If Theorem 2.3 applies, then the "FCS" technique can be used on any rational function. □

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Note. We repeat the starred equations here to emphasize that we have Factored (at step $(*)$), Cancelled (at step $(**)$), and Substituted (at step $(***)$):

$$\begin{aligned}\lim_{v\to 2}\frac{v^3-8}{v^4-16} &= \lim_{v\to 2}\frac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)} \quad (*)\\ &= \lim_{v\to 2}\frac{(v^2+2v+4)}{(v+2)(v^2+4)} \quad (**)\\ &= \frac{(2)^2+2(2)+4}{((2)+2)((2)^2+4)} = 3/8. \quad (***)\end{aligned}$$

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Exercise 2.2.38

Exercise 2.2.38. Evaluate $\displaystyle\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$.

Solution. It turns out that this can be evaluated by the FCS method, but first we must multiply by the "conjugate" of the numerator. Consider,

$$\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} = \lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}\left(\frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}\right) \text{ multiplying by 1}$$

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&= \lim_{x\to -1} \frac{\left(\sqrt{x^2+8}\right)^2-(3)^2}{(x+1)(\sqrt{x^2+8}+3)} = \lim_{x\to -1} \frac{(x^2+8)-9}{(x+1)(\sqrt{x^2+8}+3)} \\
&= \lim_{x\to -1} \frac{x^2-1}{(x+1)(\sqrt{x^2+8}+3)} \\
&= \lim_{x\to -1} \frac{(x+1)(x-1)}{(x+1)(\sqrt{x^2+8}+3)} \text{ factoring}
\end{aligned}$$

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Exercise 2.2.38 (continued)

Exercise 2.2.38. Evaluate $\displaystyle\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$.

Solution (continued).

$$\begin{aligned}
\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} &= \lim_{x\to -1} \frac{x-1}{\sqrt{x^2+8}+3} \\
&= \frac{\lim_{x\to -1} x-1}{\lim_{x\to -1} \sqrt{x^2+8}+3} \text{ by the Quotient Rule (Theorem 3.1(5))} \\
&= \frac{(-1)-1}{\sqrt{(-1)^2+8}+3} \text{ substituting by Theorem 2.2 and the Root Rule (Theorem 3.1(7))} \\
&= \frac{-2}{6} = \boxed{\frac{-1}{3}}. \quad \square
\end{aligned}$$

Exercise 2.2.38 (continued)

Exercise 2.2.38. Evaluate $\displaystyle\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$.

Solution (continued).

$$\begin{aligned}
\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} &= \lim_{x\to -1} \frac{x-1}{\sqrt{x^2+8}+3} \\
&= \frac{\lim_{x\to -1} x-1}{\lim_{x\to -1} \sqrt{x^2+8}+3} \text{ by the Quotient Rule} \\
&\quad \text{(Theorem 3.1(5))} \\
&= \frac{(-1)-1}{\sqrt{(-1)^2+8}+3} \text{ substituting by Theorem 2.2} \\
&\quad \text{and the Root Rule (Theorem 3.1(7))} \\
&= \frac{-2}{6} = \boxed{\frac{-1}{3}}. \quad \square
\end{aligned}$$

Example 2.2.11(a)(b)

Example 2.2.11. Use the Sandwich Theorem to show that:

(a) $\lim_{\theta \to 0} \sin\theta = 0$.

(b) $\lim_{\theta \to 0} \cos\theta = 1$.

Solution. In Section 1.3 we saw that $-|\theta| \leq \sin\theta \leq |\theta|$ and $0 \leq 1 - \cos\theta \leq |\theta|$ for all θ. See Figure 2.14.

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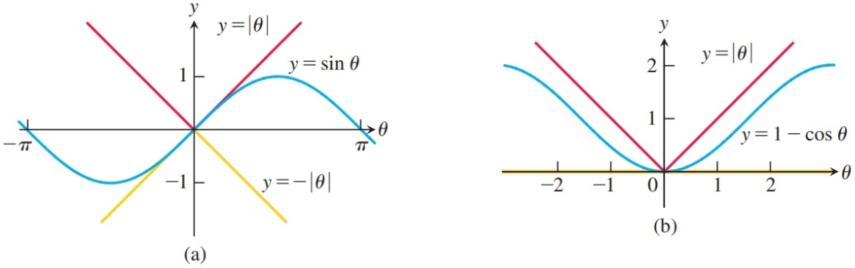

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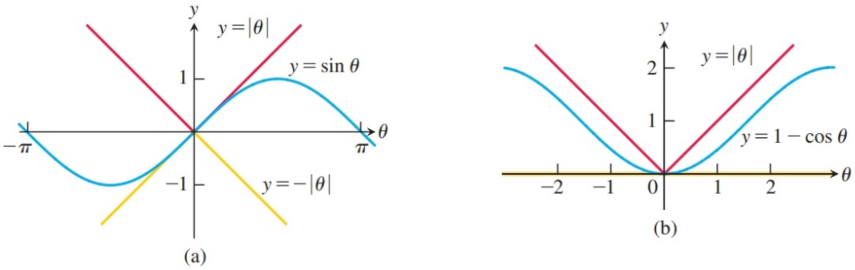
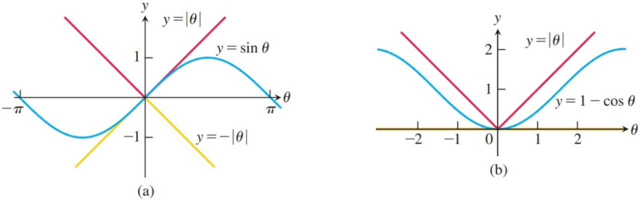
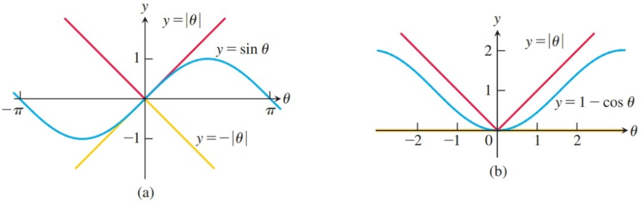


Example 2.2.11(a)(b) (continued 1)



Solution (continued). (a) We take $g(\theta) = -|\theta|$, $f(\theta) = \sin\theta$, and $h(\theta) = |\theta|$. Then there is an open interval containing $c = 0$ such that $g(\theta) \leq f(\theta) \leq h(\theta)$ (namely, the interval $(-\infty, \infty)$), except possibly at $c = 0$ itself. By Dr. Bob's Anthropomorphic Definition of Limit, $\lim_{\theta\to 0} g(\theta) = \lim_{\theta\to 0} -|\theta| = 0 = L$ and $\lim_{\theta\to 0} h(\theta) = \lim_{\theta\to 0} |\theta| = 0 = L$. So by the Sandwich Theorem (Theorem 2.4), $\lim_{\theta\to 0} f(\theta) = \lim_{\theta\to 0} \sin\theta = L = \boxed{0}$. $\square$

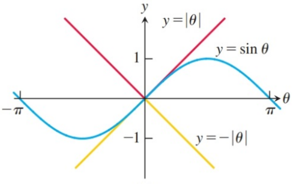
Example 2.2.11(a)(b) (continued 2)



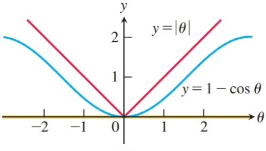
(a) (b)

Solution (continued). (b) We take $g(\theta) = 0$, $f(\theta) = 1 - \cos\theta$, and $h(\theta) = |\theta|$. Then there is an open interval containing $c = 0$ such that $g(\theta) \leq f(\theta) \leq h(\theta)$ (namely, the interval $(-\infty, \infty)$), except possibly at $c = 0$ itself. By Dr. Bob's Anthropomorphic Definition of Limit, $\lim_{\theta\to 0} g(\theta) = \lim_{\theta\to 0} 0 = 0 = L$ and $\lim_{\theta\to 0} h(\theta) = \lim_{\theta\to 0} |\theta| = 0 = L$. So by the Sandwich Theorem (Theorem 2.4), $\lim_{\theta\to 0} f(\theta) = \lim_{\theta\to 0}(1 - \cos\theta) = L = 0$. By the Difference Rule (Theorem 2.1(2)), $\lim_{\theta\to 0}(1 - \cos\theta) = 1 - \lim_{\theta\to 0} \cos\theta = 0$, so $\lim_{\theta\to 0} \cos\theta = \boxed{1}$. □

Example 2.2.11(a)(b) (continued 2)



(a)



(b)

Solution (continued). (b) We take $g(\theta) = 0$, $f(\theta) = 1 - \cos\theta$, and $h(\theta) = |\theta|$. Then there is an open interval containing $c = 0$ such that $g(\theta) \le f(\theta) \le h(\theta)$ (namely, the interval $(-\infty, \infty)$), except possibly at $c = 0$ itself. By Dr. Bob's Anthropomorphic Definition of Limit, $\lim_{\theta\to 0} g(\theta) = \lim_{\theta\to 0} 0 = 0 = L$ and $\lim_{\theta\to 0} h(\theta) = \lim_{\theta\to 0} |\theta| = 0 = L$. So by the Sandwich Theorem (Theorem 2.4), $\lim_{\theta\to 0} f(\theta) = \lim_{\theta\to 0}(1 - \cos\theta) = L = 0$. By the Difference Rule (Theorem 2.1(2)), $\lim_{\theta\to 0}(1 - \cos\theta) = 1 - \lim_{\theta\to 0}\cos\theta = 0$, so $\lim_{\theta\to 0}\cos\theta = \boxed{1}$. □

Exercise 2.2.66(a)

Exercise 2.2.66(a). Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero (that is, this holds on some open interval containing zero, except at zero itself; we will see in Section 10.9 that this does in fact hold). What, if anything, does this tell you about $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$? Give reasons for your answer.

Solution. We let $g(x) = \frac{1}{2} - \frac{x^2}{24}$, $f(x) = \frac{1 - \cos x}{x^2}$, $h(x) = \frac{1}{2}$, and $c = 0$. Then we have $g(x) \le f(x) \le h(x)$ on some open interval containing $c = 0$ except at $c = 0$ itself, by hypothesis.

Exercise 2.2.66(a)

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Solution. We let $g(x) = \frac{1}{2} - \frac{x^2}{24}$, $f(x) = \frac{1 - \cos x}{x^2}$, $h(x) = \frac{1}{2}$, and $c = 0$. Then we have $g(x) \leq f(x) \leq h(x)$ on some open interval containing $c = 0$ except at $c = 0$ itself, by hypothesis. Now $\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{1}{2} - \frac{x^2}{24} = \frac{1}{2} - \frac{0^2}{24} = \frac{1}{2}$ by Theorem 2.2, and $h(x) = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2}$ by Note 2.2.A.

Exercise 2.2.66(a)

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$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero (that is, this holds on some open interval containing zero, except at zero itself; we will see in Section 10.9 that this does in fact hold). What, if anything, does this tell you about $\lim_{x\to 0} \frac{1-\cos x}{x^2}$? Give reasons for your answer.

Solution. We let $g(x) = \frac{1}{2} - \frac{x^2}{24}$, $f(x) = \frac{1-\cos x}{x^2}$, $h(x) = \frac{1}{2}$, and $c = 0$. Then we have $g(x) \le f(x) \le h(x)$ on some open interval containing $c = 0$ except at $c = 0$ itself, by hypothesis. Now $\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{1}{2} - \frac{x^2}{24} = \frac{1}{2} - \frac{0^2}{24} = \frac{1}{2}$ by Theorem 2.2, and $h(x) = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$ by Note 2.2.A.

Exercise 2.2.66(a) (continued)

Exercise 2.2.66(a). Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero (that is, this holds on some open interval containing zero, except at zero itself; we will see in Section 10.9 that this does in fact hold). What, if anything, does this tell you about $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$? Give reasons for our answer.

Solution (continued). That is,

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} h(x) = L = 1/2.$$

Therefore, by the Sandwich Theorem (Theorem 2.4),

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1 - \cos x}{x^2} = L = \boxed{1/2}.$$

□

Exercise 2.2.79

Exercise 2.2.79. If $\lim\limits_{x\to 4} \dfrac{f(x)-5}{x-2} = 1$, then find $\lim_{x\to 4} f(x)$.

Solution. We have

$$\lim_{x\to 4} \frac{f(x)-5}{x-2} = \frac{\lim_{x\to 4}(f(x)-5)}{\lim_{x\to 4}(x-2)} \text{ by the Quotient Rule, Theorem 2.1(5)}$$

$$= \frac{\lim_{x\to 4} f(x) - \lim_{x\to 4}(5)}{\lim_{x\to 4}(x-2)} \text{ by the Difference Rule,}$$

Theorem 2.1(2)

Exercise 2.2.79

Exercise 2.2.79. If $\lim_{x\to 4} \dfrac{f(x)-5}{x-2} = 1$, then find $\lim_{x\to 4} f(x)$.

Solution. We have

$$\begin{aligned}
\lim_{x\to 4} \frac{f(x)-5}{x-2} &= \frac{\lim_{x\to 4}(f(x)-5)}{\lim_{x\to 4}(x-2)} \text{ by the Quotient Rule, Theorem 2.1(5)} \\
&= \frac{\lim_{x\to 4} f(x) - \lim_{x\to 4}(5)}{\lim_{x\to 4}(x-2)} \text{ by the Difference Rule,} \\
&\quad \text{Theorem 2.1(2)} \\
&= \frac{\lim_{x\to 4}(f(x)) - 5}{(4)-2} \text{ by Theorem 2.2} \\
&= \frac{1}{2}\lim_{x\to 4}(f(x)) - \frac{5}{2}.
\end{aligned}$$

Exercise 2.2.79

Exercise 2.2.79. If $\lim_{x\to 4} \dfrac{f(x)-5}{x-2} = 1$, then find $\lim_{x\to 4} f(x)$.

Solution. We have

$$\begin{aligned}
\lim_{x\to 4} \frac{f(x)-5}{x-2} &= \frac{\lim_{x\to 4}(f(x)-5)}{\lim_{x\to 4}(x-2)} \text{ by the Quotient Rule, Theorem 2.1(5)} \\
&= \frac{\lim_{x\to 4} f(x) - \lim_{x\to 4}(5)}{\lim_{x\to 4}(x-2)} \text{ by the Difference Rule,} \\
&\quad \text{Theorem 2.1(2)} \\
&= \frac{\lim_{x\to 4}(f(x)) - 5}{(4)-2} \text{ by Theorem 2.2} \\
&= \frac{1}{2}\lim_{x\to 4}(f(x)) - \frac{5}{2}.
\end{aligned}$$

Exercise 2.2.79 (continued)

Exercise 2.2.79. If $\lim_{x \to 4} \frac{f(x) - 5}{x - 2} = 1$, then find $\lim_{x \to 4} f(x)$.

Solution (continued). Since $\lim_{x \to 4} \frac{f(x) - 5}{x - 2} = 1$ by hypothesis, then $\frac{1}{2} \lim_{x \to 4} (f(x)) - \frac{5}{2} = 1$, or $\lim_{x \to 4} (f(x)) - 5 = 2$, and so $\boxed{\lim_{x \to 4} f(x) = 7}$. □