Calculus 1

Chapter 2. Limits and Continuity

2.3. The Precise Definition of a Limit—Examples and Proofs

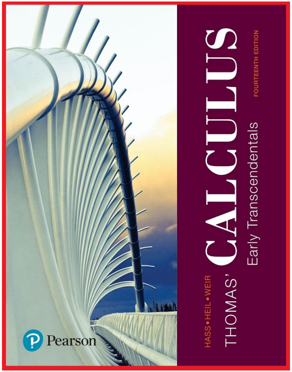


Table of contents

Example 2.3.1

Example 2.3.1. Consider the function $y = 2x - 1$ near $x = 4$. Intuitively it seems clear that y is close to 7 when x is close to 4, so $\lim_{x\to 4}(2x - 1) = 7$. However, how close to 4 does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

Solution. We use absolute value to measure distance, so the distance between x and 4 is $|x - 4|$, and the distance between y and 7 is $|y - 7|$. So the question has become: How small must $|x - 4|$ be so that $|y - 7| < 2$?

Example 2.3.1

Example 2.3.1. Consider the function $y = 2x - 1$ near $x = 4$. Intuitively it seems clear that y is close to 7 when x is close to 4, so $\lim_{x \to 4}(2x - 1) = 7$. However, how close to 4 does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

Solution. We use absolute value to measure distance, so the distance between x and 4 is $|x - 4|$, and the distance between y and 7 is $|y - 7|$. So the question has become: How small must $|x - 4|$ be so that $|y - 7| < 2$? Since $y = 2x - 1$, we consider the desired inequality $|y - 7| = |(2x - 1) - 7| < 2$. This is equivalent to $|2x - 8| < 2$ or $|2(x - 4)| < 2$ or $|x - 4| < 1$.

Example 2.3.1

Example 2.3.1. Consider the function $y = 2x - 1$ near $x = 4$. Intuitively it seems clear that y is close to 7 when x is close to 4, so $\lim_{x\to 4}(2x - 1) = 7$. However, how close to 4 does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

Solution. We use absolute value to measure distance, so the distance between x and 4 is $|x - 4|$, and the distance between y and 7 is $|y - 7|$. So the question has become: How small must $|x - 4|$ be so that $|y - 7| < 2$? Since $y = 2x - 1$, we consider the desired inequality $|y - 7| = |(2x - 1) - 7| < 2$. This is equivalent to $|2x - 8| < 2$ or $|2(x - 4)| < 2$ or $|x - 4| < 1$. Hence, we see that to get the distance between $y = 2x - 1$ and 7 to be less that 2, the distance between x and 4 must be less than 1. We now consider the graph of the function in light of this information.

Example 2.3.1

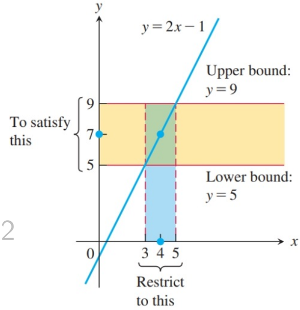
Example 2.3.1. Consider the function $y = 2x - 1$ near $x = 4$. Intuitively it seems clear that y is close to 7 when x is close to 4, so $\lim_{x\to 4}(2x-1) = 7$. However, how close to 4 does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

Solution. We use absolute value to measure distance, so the distance between x and 4 is $|x-4|$, and the distance between y and 7 is $|y-7|$. So the question has become: How small must $|x-4|$ be so that $|y-7| < 2$? Since $y = 2x-1$, we consider the desired inequality $|y-7| = |(2x-1)-7| < 2$. This is equivalent to $|2x-8| < 2$ or $|2(x-4)| < 2$ or $|x-4| < 1$. Hence, we see that to get the distance between $y = 2x-1$ and 7 to be less that 2, $\boxed{\text{the distance between } x \text{ and 4 must be less than 1.}}$ We now consider the graph of the function in light of this information.

Example 2.3.1 (continued)

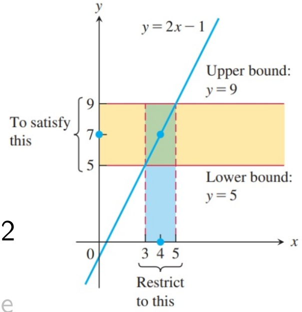
Solution (continued). Notice that $|x-4|<1$ means $-1<x-4<1$ or $3<x<5$. The inequality $|y-7|<2$ means $-2<y-7<2$ or $5<y<9$. We have the following graph (Figure 2.15):

So we have $|y-7|<2$ (i.e., the graph of the function lies in the horizontal yellow band) when $|x-4|<1$ (i.e., when the graph of the function lies in the vertical blue band). Notice that if the yellow band is made smaller then the blue band must also be made smaller (for this function, the blue band must be at most $1/2$ times the width of the yellow band).



Example 2.3.1 (continued)

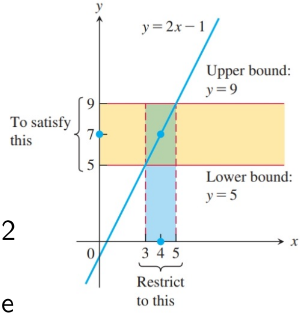
Solution (continued). Notice that $|x-4|<1$ means $-1<x-4<1$ or $3<x<5$. The inequality $|y-7|<2$ means $-2<y-7<2$ or $5<y<9$. We have the following graph (Figure 2.15):



So we have $|y-7|<2$ (i.e., the graph of the function lies in the horizontal yellow band) when $|x-4|<1$ (i.e., when the graph of the function lies in the vertical blue band). Notice that if the yellow band is made smaller then the blue band must also be made smaller (for this function, the blue band must be at most $1/2$ times the width of the yellow band). With the yellow band really narrow, the blue band must be really narrow and the green box where they intersect must be really small; in fact, the green box then closes in around the point $(4,7)$. Notice also that the graph of the function tries to pass through this point (and it succeeds). $\square$

Example 2.3.1 (continued)

Solution (continued). Notice that $|x-4|<1$ means $-1<x-4<1$ or $3<x<5$. The inequality $|y-7|<2$ means $-2<y-7<2$ or $5<y<9$. We have the following graph (Figure 2.15):



So we have $|y-7|<2$ (i.e., the graph of the function lies in the horizontal yellow band) when $|x-4|<1$ (i.e., when the graph of the function lies in the vertical blue band). Notice that if the yellow band is made smaller then the blue band must also be made smaller (for this function, the blue band must be at most $1/2$ times the width of the yellow band). With the yellow band really narrow, the blue band must be really narrow and the green box where they intersect must be really small; in fact, the green box then closes in around the point $(4,7)$. Notice also that the graph of the function tries to pass through this point (and it succeeds). $\square$

Exercise 2.3.A

Example 2.3.A. Prove for $f(x) = mx + b$, $m \neq 0$, that $\lim_{x \to a} f(x) = f(a)$.

Proof. In the notation of the definition of limit, we have $f(x) = mx + b$ where $m \neq 0$, $c = a$, and we claim $L = f(a) = ma + b$. Notice that f is defined on all of $\mathbb{R}$, so f is defined on an open interval about a (namely, the interval $(-\infty, \infty)$) as required by the definition.

Exercise 2.3.A

Example 2.3.A. Prove for $f(x) = mx + b$, $m \neq 0$, that $\lim_{x \to a} f(x) = f(a)$.

Proof. In the notation of the definition of limit, we have $f(x) = mx + b$ where $m \neq 0$, $c = a$, and we claim $L = f(a) = ma + b$. Notice that f is defined on all of $\mathbb{R}$, so f is defined on an open interval about a (namely, the interval $(-\infty, \infty)$) as required by the definition.

Next, let $\varepsilon > 0$. We need to find $\delta > 0$ satisfying the conditions of the definition of limit. We expect δ to depend in some way on ε.

Exercise 2.3.A

Example 2.3.A. Prove for $f(x) = mx + b$, $m \neq 0$, that $\lim_{x \to a} f(x) = f(a)$.

Proof. In the notation of the definition of limit, we have $f(x) = mx + b$ where $m \neq 0$, $c = a$, and we claim $L = f(a) = ma + b$. Notice that f is defined on all of $\mathbb{R}$, so f is defined on an open interval about a (namely, the interval $(-\infty, \infty)$) as required by the definition.

Next, let $\varepsilon > 0$. We need to find $\delta > 0$ satisfying the conditions of the definition of limit. We expect δ to depend in some way on ε.

We choose $\delta = \varepsilon/|m| > 0$ (for now, never mind where this comes from; Example 2.3.2 illustrates the idea behind this choice in the case that $m = 5$); this is where we use the fact that $m \neq 0$.

Exercise 2.3.A

Example 2.3.A. Prove for $f(x) = mx + b$, $m \neq 0$, that $\lim_{x \to a} f(x) = f(a)$.

Proof. In the notation of the definition of limit, we have $f(x) = mx + b$ where $m \neq 0$, $c = a$, and we claim $L = f(a) = ma + b$. Notice that f is defined on all of $\mathbb{R}$, so f is defined on an open interval about a (namely, the interval $(-\infty, \infty)$) as required by the definition.

Next, let $\varepsilon > 0$. We need to find $\delta > 0$ satisfying the conditions of the definition of limit. We expect δ to depend in some way on ε.

We choose $\delta = \varepsilon/|m| > 0$ (for now, never mind where this comes from; Example 2.3.2 illustrates the idea behind this choice in the case that $m = 5$); this is where we use the fact that $m \neq 0$. We want to show that $0 < |x - c| = |x - a| < \delta$ implies $|f(x) - L| = |(mx + b) - (ma + b)| < \varepsilon$.

Exercise 2.3.A

Example 2.3.A. Prove for $f(x) = mx + b$, $m \neq 0$, that $\lim_{x \to a} f(x) = f(a)$.

Proof. In the notation of the definition of limit, we have $f(x) = mx + b$ where $m \neq 0$, $c = a$, and we claim $L = f(a) = ma + b$. Notice that f is defined on all of $\mathbb{R}$, so f is defined on an open interval about a (namely, the interval $(-\infty, \infty)$) as required by the definition.

Next, let $\varepsilon > 0$. We need to find $\delta > 0$ satisfying the conditions of the definition of limit. We expect δ to depend in some way on ε.

We choose $\delta = \varepsilon/|m| > 0$ (for now, never mind where this comes from; Example 2.3.2 illustrates the idea behind this choice in the case that $m = 5$); this is where we use the fact that $m \neq 0$. We want to show that $0 < |x - c| = |x - a| < \delta$ implies $|f(x) - L| = |(mx + b) - (ma + b)| < \varepsilon$.

Exercise 2.3.A (continued)

Example 2.3.A. Prove for $f(x) = mx + b$, $m \neq 0$, that $\lim_{x \to a} f(x) = f(a)$.

Proof (continued). Suppose $0 < |x - a| < \delta$, or equivalently (given our choice of δ) $0 < |x - a| < \varepsilon/|m|$. Multiplying through by $|m|$, this implies that $0 < |m|\,|x - a| < \varepsilon$ which in turn implies $|m|\,|x - a| < \varepsilon$ or $|m(x - a)| < \varepsilon$ or $|mx - ma| < \varepsilon$ or $|mx + (b - b) - ma| < \varepsilon$ or $|(mx + b) - (ma + b)| < \varepsilon$ or $|f(x) - f(a)| < \varepsilon$, as desired.

Exercise 2.3.A (continued)

Example 2.3.A. Prove for $f(x) = mx + b$, $m \neq 0$, that $\lim_{x \to a} f(x) = f(a)$.

Proof (continued). Suppose $0 < |x - a| < \delta$, or equivalently (given our choice of δ) $0 < |x - a| < \varepsilon/|m|$. Multiplying through by $|m|$, this implies that $0 < |m|\,|x - a| < \varepsilon$ which in turn implies $|m|\,|x - a| < \varepsilon$ or $|m(x - a)| < \varepsilon$ or $|mx - ma| < \varepsilon$ or $|mx + (b - b) - ma| < \varepsilon$ or $|(mx + b) - (ma + b)| < \varepsilon$ or $|f(x) - f(a)| < \varepsilon$, as desired. Therefore, by the definition of limit, $\lim_{x \to a} f(x) = f(a)$. □

Exercise 2.3.A (continued)

Example 2.3.A. Prove for $f(x) = mx + b$, $m \neq 0$, that $\lim_{x \to a} f(x) = f(a)$.

Proof (continued). Suppose $0 < |x - a| < \delta$, or equivalently (given our choice of δ) $0 < |x - a| < \varepsilon/|m|$. Multiplying through by $|m|$, this implies that $0 < |m|\,|x - a| < \varepsilon$ which in turn implies $|m|\,|x - a| < \varepsilon$ or $|m(x - a)| < \varepsilon$ or $|mx - ma| < \varepsilon$ or $|mx + (b - b) - ma| < \varepsilon$ or $|(mx + b) - (ma + b)| < \varepsilon$ or $|f(x) - f(a)| < \varepsilon$, as desired. Therefore, by the definition of limit, $\lim_{x \to a} f(x) = f(a)$. $\square$

Note. Notice the logic of being given an arbitrary $\varepsilon > 0$ first, and *then* having to *find* (based on ε) a $\delta > 0$ such that the definition of limit is satisfied.

Exercise 2.3.A (continued)

Example 2.3.A. Prove for $f(x) = mx + b$, $m \neq 0$, that $\lim_{x \to a} f(x) = f(a)$.

Proof (continued). Suppose $0 < |x - a| < \delta$, or equivalently (given our choice of δ) $0 < |x - a| < \varepsilon/|m|$. Multiplying through by $|m|$, this implies that $0 < |m|\,|x - a| < \varepsilon$ which in turn implies $|m|\,|x - a| < \varepsilon$ or $|m(x - a)| < \varepsilon$ or $|mx - ma| < \varepsilon$ or $|mx + (b - b) - ma| < \varepsilon$ or $|(mx + b) - (ma + b)| < \varepsilon$ or $|f(x) - f(a)| < \varepsilon$, as desired. Therefore, by the definition of limit, $\lim_{x \to a} f(x) = f(a)$. $\square$

Note. Notice the logic of being given an arbitrary $\varepsilon > 0$ first, and *then* having to *find* (based on ε) a $\delta > 0$ such that the definition of limit is satisfied.

Exercise 2.3.3

Example 2.3.3. Use the formal definition of limit to prove:
(a) $\lim_{x \to c} x = c$, **(b)** $\lim_{x \to c} k = k$ where k is a constant.

Proof. **(a)** We have $f(x) = x$ and we claim $L = c$. Notice that f is defined on all of $\mathbb{R}$, so f is defined on an open interval about c (namely, the interval $(-\infty, \infty)$) as required by the definition.

Exercise 2.3.3

Example 2.3.3. Use the formal definition of limit to prove:
(a) $\lim_{x\to c} x = c$, **(b)** $\lim_{x\to c} k = k$ where k is a constant.

Proof. (a) We have $f(x) = x$ and we claim $L = c$. Notice that f is defined on all of $\mathbb{R}$, so f is defined on an open interval about c (namely, the interval $(-\infty, \infty)$) as required by the definition.

Next, let $\varepsilon > 0$. We choose $\delta = \varepsilon > 0$.

Exercise 2.3.3

Example 2.3.3. Use the formal definition of limit to prove:
(a) $\lim_{x\to c} x = c$, **(b)** $\lim_{x\to c} k = k$ where k is a constant.

Proof. (a) We have $f(x) = x$ and we claim $L = c$. Notice that f is defined on all of $\mathbb{R}$, so f is defined on an open interval about c (namely, the interval $(-\infty, \infty)$) as required by the definition.

Next, let $\varepsilon > 0$. We choose $\delta = \varepsilon > 0$.

Suppose $0 < |x - c| < \delta$, or equivalently (given our choice of δ) $0 < |x - c| < \varepsilon$. This implies that $|x - c| < \varepsilon$, or $|f(x) - L| < \varepsilon$, as desired. Therefore, by the definition of limit, $\lim_{x\to c} x = c$. □

Exercise 2.3.3

Example 2.3.3. Use the formal definition of limit to prove:
(a) $\lim_{x \to c} x = c$, **(b)** $\lim_{x \to c} k = k$ where k is a constant.

Proof. (a) We have $f(x) = x$ and we claim $L = c$. Notice that f is defined on all of $\mathbb{R}$, so f is defined on an open interval about c (namely, the interval $(-\infty, \infty)$) as required by the definition.

Next, let $\varepsilon > 0$. We choose $\delta = \varepsilon > 0$.

Suppose $0 < |x - c| < \delta$, or equivalently (given our choice of δ) $0 < |x - c| < \varepsilon$. This implies that $|x - c| < \varepsilon$, or $|f(x) - L| < \varepsilon$, as desired. Therefore, by the definition of limit, $\lim_{x \to c} x = c$. $\square$

Exercise 2.3.3 (continued 1)

Note. In part **(a)**, we chose $\delta = \varepsilon$ but we could have chosen $\delta > 0$ to be any value between 0 and ε. We would still have $|f(x) - L| = |x - c| < \delta \leq \varepsilon$. This is illustrated in Figure 2.19:

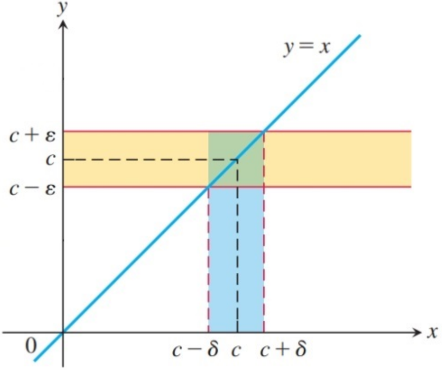


Figure 2.19 (modified)

Exercise 2.3.3 (continued 2)

Example 2.3.3. Use the formal definition of limit to prove:
(a) $\lim_{x \to c} x = c$, **(b)** $\lim_{x \to c} k = k$ where k is a constant.

Proof (continued). **(b)** We have $f(x) = k$ and we claim $L = k$. Notice that f is defined on all of $\mathbb{R}$, so f is defined on an open interval about c (namely, the interval $(-\infty, \infty)$) as required by the definition.

Exercise 2.3.3 (continued 2)

Example 2.3.3. Use the formal definition of limit to prove:
(a) $\lim_{x\to c} x = c$, **(b)** $\lim_{x\to c} k = k$ where k is a constant.

Proof (continued). (b) We have $f(x) = k$ and we claim $L = k$. Notice that f is defined on all of $\mathbb{R}$, so f is defined on an open interval about c (namely, the interval $(-\infty, \infty)$) as required by the definition.

Next, let $\varepsilon > 0$. We choose $\delta = \varepsilon > 0$.

Exercise 2.3.3 (continued 2)

Example 2.3.3. Use the formal definition of limit to prove:
(a) $\lim_{x \to c} x = c$, **(b)** $\lim_{x \to c} k = k$ where k is a constant.

Proof (continued). (b) We have $f(x) = k$ and we claim $L = k$. Notice that f is defined on all of $\mathbb{R}$, so f is defined on an open interval about c (namely, the interval $(-\infty, \infty)$) as required by the definition.

Next, let $\varepsilon > 0$. We choose $\delta = \varepsilon > 0$.

Suppose $0 < |x - c| < \delta$. Then $|k - k| = 0 < \varepsilon$ or $|f(x) - L| < \varepsilon$, as desired. Therefore, by the definition of limit, $\lim_{x \to c} k = k$. □

Exercise 2.3.3 (continued 2)

Example 2.3.3. Use the formal definition of limit to prove:
(a) $\lim_{x\to c} x = c$, **(b)** $\lim_{x\to c} k = k$ where k is a constant.

Proof (continued). (b) We have $f(x) = k$ and we claim $L = k$. Notice that f is defined on all of $\mathbb{R}$, so f is defined on an open interval about c (namely, the interval $(-\infty, \infty)$) as required by the definition.

Next, let $\varepsilon > 0$. We choose $\delta = \varepsilon > 0$.

Suppose $0 < |x - c| < \delta$. Then $|k - k| = 0 < \varepsilon$ or $|f(x) - L| < \varepsilon$, as desired. Therefore, by the definition of limit, $\lim_{x\to c} k = k$. $\square$

Exercise 2.3.3 (continued 3)

Note. In part **(b)**, we have $|f(x) - L| = |k - k| = 0 < \varepsilon$, regardless of the choice of δ so that we could have chosen $\delta > 0$ as any value. This property is unique to constant functions. This is illustrated in Figure 2.20:

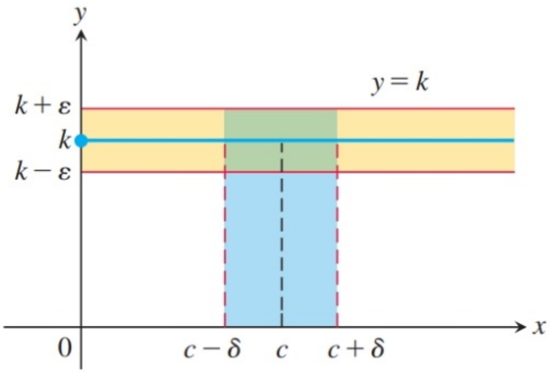


Figure 2.20

Example 2.3.4

Example 2.3.4. For the limit $\lim_{x\to 5} \sqrt{x-1} = 2$ (true by the Root Rule, Theorem 2.1(7)), find $\delta > 0$ that works for $\varepsilon = 1$. That is, find a $\delta > 0$ such that

$$0 < |x-5| < \delta \text{ implies } |\sqrt{x-1} - 2| < 1.$$

Solution. The graph of the relevant part of $y = f(x) = \sqrt{x-1}$ is given in Figure 2.22:

Example 2.3.4

Example 2.3.4. For the limit $\lim_{x\to 5}\sqrt{x-1}=2$ (true by the Root Rule, Theorem 2.1(7)), find $\delta>0$ that works for $\varepsilon=1$. That is, find a $\delta>0$ such that

$$0<|x-5|<\delta \text{ implies } |\sqrt{x-1}-2|<1.$$

Solution. The graph of the relevant part of $y=f(x)=\sqrt{x-1}$ is given in Figure 2.22:

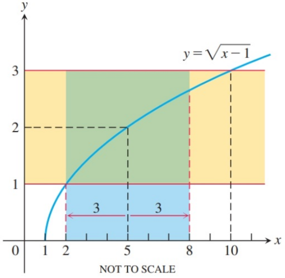


Figure 2.22

In the notation of the definition of limit, we have $f(x)=\sqrt{x-1}$, $c=5$, and we claim $L=2$. To get $|f(x)-L|<\varepsilon$, or $|\sqrt{x-1}-2|<1$, we need the graph of $y=f(x)$ to lie in the yellow band. We see that this occurs for x between 2 and 10. Since we measure the distance of x from 5, we see that we can go 3 units to the left and 5 units to the right of 5. We let δ be the smaller of these two distances. So we take $\delta=3$.

Example 2.3.4

Example 2.3.4. For the limit $\lim_{x \to 5} \sqrt{x-1} = 2$ (true by the Root Rule, Theorem 2.1(7)), find $\delta > 0$ that works for $\varepsilon = 1$. That is, find a $\delta > 0$ such that

$$0 < |x - 5| < \delta \text{ implies } |\sqrt{x-1} - 2| < 1.$$

Solution. The graph of the relevant part of $y = f(x) = \sqrt{x-1}$ is given in Figure 2.22:

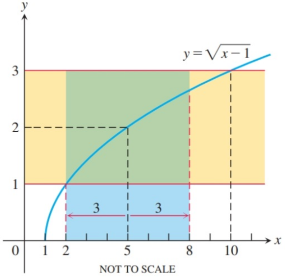
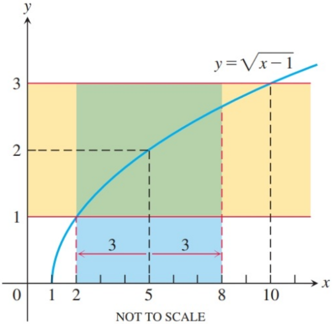


Figure 2.22

In the notation of the definition of limit, we have $f(x) = \sqrt{x-1}$, $c = 5$, and we claim $L = 2$. To get $|f(x) - L| < \varepsilon$, or $|\sqrt{x-1} - 2| < 1$, we need the graph of $y = f(x)$ to lie in the yellow band. We see that this occurs for x between 2 and 10. Since we measure the distance of x from 5, we see that we can go 3 units to the left and 5 units to the right of 5. We let δ be the smaller of these two distances. So we take $\delta = 3$.

Example 2.3.4 (continued)



Notice that if $0 < |x - 5| < \delta = 3$, then x lies in the blue vertical band. So the corresponding function values lie in the horizontal yellow band. Notice that the graph $y = f(x)$ then intersects the vertical edges of the resulting green box and does not intersect the horizontal edges (except possibly at a corner). So the chosen δ value of 3 yields the desired behavior:

$$0 < |x - 5| < \delta \text{ implies } |\sqrt{x-1} - 2| < 1. \qquad \square$$

Exercise 2.3.20

Exercise 2.3.20. Consider $f(x) = \sqrt{x-7}$, $c = 23$, $\varepsilon = 1$, and $L = 4$. Find an open interval about c on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, the inequality $|f(x) - L| < \varepsilon$ holds.

Solution. First, notice that the domain of $f(x) = \sqrt{x-7}$ is $x \geq 7$. Applying Step 1 of the previous note, we solve the inequality $|f(x) - L| < \varepsilon$, or $|\sqrt{x-7} - 4| < 1$. This is equivalent to $-1 < \sqrt{x-7} - 4 < 1$ or $3 < \sqrt{x-7} < 5$ or (since the squaring function is an increasing function for positive inputs) $3^2 < (\sqrt{x-7})^2 < 5^2$ or $9 < x - 7 < 25$ or $16 < x < 32$. So an open interval on which the inequality holds is $\boxed{(16, 32)}$.

Exercise 2.3.20

Exercise 2.3.20. Consider $f(x) = \sqrt{x-7}$, $c = 23$, $\varepsilon = 1$, and $L = 4$. Find an open interval about c on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, the inequality $|f(x) - L| < \varepsilon$ holds.

Solution. First, notice that the domain of $f(x) = \sqrt{x-7}$ is $x \geq 7$. Applying Step 1 of the previous note, we solve the inequality $|f(x) - L| < \varepsilon$, or $|\sqrt{x-7} - 4| < 1$. This is equivalent to $-1 < \sqrt{x-7} - 4 < 1$ or $3 < \sqrt{x-7} < 5$ or (since the squaring function is an increasing function for positive inputs) $3^2 < (\sqrt{x-7})^2 < 5^2$ or $9 < x - 7 < 25$ or $16 < x < 32$. So an open interval on which the inequality holds is $\boxed{(16, 32)}$.

Now the distance from $c = 23$ to 16 is $\delta_1 = 7$, and the distance from $c = 23$ to 32 is $\delta_2 = 9$. So we choose δ as the smaller of δ_1 and δ_2; that is, we take $\boxed{\delta = 7}$. Then for $0 < |x - c| = |x - 23| < \delta = 7$, we have $x \in (16, 30) \subset (16, 32)$ and so the inequality holds, as desired. $\square$

Exercise 2.3.20

Exercise 2.3.20. Consider $f(x) = \sqrt{x-7}$, $c = 23$, $\varepsilon = 1$, and $L = 4$. Find an open interval about c on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, the inequality $|f(x) - L| < \varepsilon$ holds.

Solution. First, notice that the domain of $f(x) = \sqrt{x-7}$ is $x \geq 7$. Applying Step 1 of the previous note, we solve the inequality $|f(x) - L| < \varepsilon$, or $|\sqrt{x-7} - 4| < 1$. This is equivalent to $-1 < \sqrt{x-7} - 4 < 1$ or $3 < \sqrt{x-7} < 5$ or (since the squaring function is an increasing function for positive inputs) $3^2 < (\sqrt{x-7})^2 < 5^2$ or $9 < x - 7 < 25$ or $16 < x < 32$. So an open interval on which the inequality holds is $\boxed{(16, 32)}$.

Now the distance from $c = 23$ to 16 is $\delta_1 = 7$, and the distance from $c = 23$ to 32 is $\delta_2 = 9$. So we choose δ as the smaller of δ_1 and δ_2; that is, we take $\boxed{\delta = 7}$. Then for $0 < |x - c| = |x - 23| < \delta = 7$, we have $x \in (16, 30) \subset (16, 32)$ and so the inequality holds, as desired. $\square$

Exercise 2.3.40

Exercise 2.3.40. Prove that $\lim_{x\to 0} \sqrt{4-x} = 2$.

Proof. We use the formal definition of limit. We have $f(x) = \sqrt{4-x}$, $c = 0$, and we claim $L = 2$. The domain of f is $x \leq 4$, so f is defined on an open interval containing $c = 0$, say $(-\infty, 4)$. Let $\varepsilon > 0$.

Exercise 2.3.40

Exercise 2.3.40. Prove that $\lim_{x \to 0} \sqrt{4-x} = 2$.

Proof. We use the formal definition of limit. We have $f(x) = \sqrt{4-x}$, $c = 0$, and we claim $L = 2$. The domain of f is $x \leq 4$, so f is defined on an open interval containing $c = 0$, say $(-\infty, 4)$. Let $\varepsilon > 0$.

[This is not part of the proof! We step aside and look for $\delta > 0$ such that $0 < |x - c| = |x - 0| = |x| < \delta$ implies $|f(x) - L| = |\sqrt{4-x} - 2| < \varepsilon$. Now $|\sqrt{4-x} - 2| < \varepsilon$ is equivalent to $-\varepsilon < \sqrt{4-x} - 2 < \varepsilon$ or $2 - \varepsilon < \sqrt{4-x} < 2 + \varepsilon$ or (since the squaring function is an increasing function for positive inputs) $(2-\varepsilon)^2 < (\sqrt{4-x})^2 < (2+\varepsilon)^2$ (where $0 < \varepsilon < 2$) or $4 - 4\varepsilon + \varepsilon^2 < 4 - x < 4 + 4\varepsilon + \varepsilon^2$ or $-4\varepsilon + \varepsilon^2 < -x < 4\varepsilon + \varepsilon^2$ or $4\varepsilon - \varepsilon^2 > x > -4\varepsilon - \varepsilon^2$ or $-(4\varepsilon + \varepsilon^2) < x < 4\varepsilon - \varepsilon^2$.

Exercise 2.3.40

Exercise 2.3.40. Prove that $\lim_{x\to 0} \sqrt{4-x} = 2$.

Proof. We use the formal definition of limit. We have $f(x) = \sqrt{4-x}$, $c = 0$, and we claim $L = 2$. The domain of f is $x \le 4$, so f is defined on an open interval containing $c = 0$, say $(-\infty, 4)$. Let $\varepsilon > 0$.
[This is not part of the proof! We step aside and look for $\delta > 0$ such that $0 < |x - c| = |x - 0| = |x| < \delta$ implies $|f(x) - L| = |\sqrt{4-x} - 2| < \varepsilon$. Now $|\sqrt{4-x} - 2| < \varepsilon$ is equivalent to $-\varepsilon < \sqrt{4-x} - 2 < \varepsilon$ or $2 - \varepsilon < \sqrt{4-x} < 2 + \varepsilon$ or (since the squaring function is an increasing function for positive inputs) $(2-\varepsilon)^2 < (\sqrt{4-x})^2 < (2+\varepsilon)^2$ (where $0 < \varepsilon < 2$) or $4 - 4\varepsilon + \varepsilon^2 < 4 - x < 4 + 4\varepsilon + \varepsilon^2$ or $-4\varepsilon + \varepsilon^2 < -x < 4\varepsilon + \varepsilon^2$ or $4\varepsilon - \varepsilon^2 > x > -4\varepsilon - \varepsilon^2$ or $-(4\varepsilon + \varepsilon^2) < x < 4\varepsilon - \varepsilon^2$. So the inequality $|f(x) - L| < \varepsilon$ holds on the interval $(-(4\varepsilon + \varepsilon^2), 4\varepsilon - \varepsilon^2)$, where we need $0 < \varepsilon < 2$. Now the distance from $c = 0$ to $-(4\varepsilon + \varepsilon^2)$ is $\delta_1 = 4\varepsilon + \varepsilon^2$, and the distance from $c = 0$ to $4\varepsilon - \varepsilon^2$ is $\delta_2 = 4\varepsilon - \varepsilon^2$. We choose δ to be the smaller of δ_1 and δ_2, so we choose $\delta = \delta_2 = 4\varepsilon - \varepsilon^2$ where we need $0 < \varepsilon < 2$.]

Exercise 2.3.40

Exercise 2.3.40. Prove that $\lim_{x\to 0} \sqrt{4-x} = 2$.

Proof. We use the formal definition of limit. We have $f(x) = \sqrt{4-x}$, $c = 0$, and we claim $L = 2$. The domain of f is $x \le 4$, so f is defined on an open interval containing $c = 0$, say $(-\infty, 4)$. Let $\varepsilon > 0$.
[This is not part of the proof! We step aside and look for $\delta > 0$ such that $0 < |x - c| = |x - 0| = |x| < \delta$ implies $|f(x) - L| = |\sqrt{4-x} - 2| < \varepsilon$. Now $|\sqrt{4-x} - 2| < \varepsilon$ is equivalent to $-\varepsilon < \sqrt{4-x} - 2 < \varepsilon$ or $2 - \varepsilon < \sqrt{4-x} < 2 + \varepsilon$ or (since the squaring function is an increasing function for positive inputs) $(2-\varepsilon)^2 < (\sqrt{4-x})^2 < (2+\varepsilon)^2$ (where $0 < \varepsilon < 2$) or $4 - 4\varepsilon + \varepsilon^2 < 4 - x < 4 + 4\varepsilon + \varepsilon^2$ or $-4\varepsilon + \varepsilon^2 < -x < 4\varepsilon + \varepsilon^2$ or $4\varepsilon - \varepsilon^2 > x > -4\varepsilon - \varepsilon^2$ or $-(4\varepsilon + \varepsilon^2) < x < 4\varepsilon - \varepsilon^2$. So the inequality $|f(x) - L| < \varepsilon$ holds on the interval $(-(4\varepsilon + \varepsilon^2), 4\varepsilon - \varepsilon^2)$, where we need $0 < \varepsilon < 2$. Now the distance from $c = 0$ to $-(4\varepsilon + \varepsilon^2)$ is $\delta_1 = 4\varepsilon + \varepsilon^2$, and the distance from $c = 0$ to $4\varepsilon - \varepsilon^2$ is $\delta_2 = 4\varepsilon - \varepsilon^2$. We choose δ to be the smaller of δ_1 and δ_2, so we choose $\delta = \delta_2 = 4\varepsilon - \varepsilon^2$ where we need $0 < \varepsilon < 2$.**]**

Exercise 2.3.40 (continued)

Exercise 2.3.40. Prove that $\lim_{x\to 0}\sqrt{4-x}=2$.

Proof (continued). If $\varepsilon<2$ then choose $\delta=4\varepsilon-\varepsilon^2>0$. Suppose that $0<|x-c|=|x-0|=|x|<\delta=4\varepsilon-\varepsilon^2$.

Exercise 2.3.40 (continued)

Exercise 2.3.40. Prove that $\lim_{x\to 0} \sqrt{4-x} = 2$.

Proof (continued). If $\varepsilon < 2$ then choose $\delta = 4\varepsilon - \varepsilon^2 > 0$. Suppose that $0 < |x - c| = |x - 0| = |x| < \delta = 4\varepsilon - \varepsilon^2$. Then $-(4\varepsilon - \varepsilon^2) < x < 4\varepsilon - \varepsilon^2$ which implies $4\varepsilon - \varepsilon^2 > -x > -(4\varepsilon - \varepsilon^2)$ or $-(4\varepsilon - \varepsilon^2) < -x < 4\varepsilon - \varepsilon^2$ or $4 - 4\varepsilon + \varepsilon^2 < 4 - x < 4 + 4\varepsilon - \varepsilon^2$ or (since $4 + 4\varepsilon - \varepsilon^2 < 4 + 4\varepsilon + \varepsilon^2$) $4 - 4\varepsilon + \varepsilon^2 < 4 - x < 4 + 4\varepsilon + \varepsilon^2$ or $(2-\varepsilon)^2 < 4 - x < (2+\varepsilon)^2$ or (since the square root function is increasing) $\sqrt{(2-\varepsilon)^2} < \sqrt{4-x} < \sqrt{(2+\varepsilon)^2}$ or $|2-\varepsilon| < \sqrt{4-x} < |2+\varepsilon|$ or (since $0 < \varepsilon < 2$) $2 - \varepsilon < \sqrt{4-x} < 2 + \varepsilon$ or $-\varepsilon < \sqrt{4-x} - 2 < \varepsilon$ or $|\sqrt{4-x} - 2| < \varepsilon$ or $|f(x) - L| < \varepsilon$, as desired.

Exercise 2.3.40 (continued)

Exercise 2.3.40. Prove that $\lim_{x\to 0}\sqrt{4-x}=2$.

Proof (continued). If $\varepsilon<2$ then choose $\delta=4\varepsilon-\varepsilon^2>0$. Suppose that $0<|x-c|=|x-0|=|x|<\delta=4\varepsilon-\varepsilon^2$. Then $-(4\varepsilon-\varepsilon^2)<x<4\varepsilon-\varepsilon^2$ which implies $4\varepsilon-\varepsilon^2>-x>-(4\varepsilon-\varepsilon^2)$ or $-(4\varepsilon-\varepsilon^2)<-x<4\varepsilon-\varepsilon^2$ or $4-4\varepsilon+\varepsilon^2<4-x<4+4\varepsilon-\varepsilon^2$ or (since $4+4\varepsilon-\varepsilon^2<4+4\varepsilon+\varepsilon^2$) $4-4\varepsilon+\varepsilon^2<4-x<4+4\varepsilon+\varepsilon^2$ or $(2-\varepsilon)^2<4-x<(2+\varepsilon)^2$ or (since the square root function is increasing) $\sqrt{(2-\varepsilon)^2}<\sqrt{4-x}<\sqrt{(2+\varepsilon)^2}$ or $|2-\varepsilon|<\sqrt{4-x}<|2+\varepsilon|$ or (since $0<\varepsilon<2$) $2-\varepsilon<\sqrt{4-x}<2+\varepsilon$ or $-\varepsilon<\sqrt{4-x}-2<\varepsilon$ or $|\sqrt{4-x}-2|<\varepsilon$ or $|f(x)-L|<\varepsilon$, as desired.

If $\varepsilon\geq 2$, then choose $\delta=4(1)-(1)^2=3$. Then by the computation above, we have for $0<|x-0|<\delta=3$, we have $|f(x)-L|<1$ (we repeat the computation above with $\varepsilon=1$ to establish this). Then we also have $|f(x)-L|<1<2\leq\varepsilon$, or $|f(x)-L|<\varepsilon$, as desired. $\square$

Exercise 2.3.40 (continued)

Exercise 2.3.40. Prove that $\lim_{x\to 0} \sqrt{4-x} = 2$.

Proof (continued). If $\varepsilon < 2$ then choose $\delta = 4\varepsilon - \varepsilon^2 > 0$. Suppose that $0 < |x-c| = |x-0| = |x| < \delta = 4\varepsilon - \varepsilon^2$. Then $-(4\varepsilon - \varepsilon^2) < x < 4\varepsilon - \varepsilon^2$ which implies $4\varepsilon - \varepsilon^2 > -x > -(4\varepsilon - \varepsilon^2)$ or $-(4\varepsilon - \varepsilon^2) < -x < 4\varepsilon - \varepsilon^2$ or $4 - 4\varepsilon + \varepsilon^2 < 4 - x < 4 + 4\varepsilon - \varepsilon^2$ or (since $4 + 4\varepsilon - \varepsilon^2 < 4 + 4\varepsilon + \varepsilon^2$) $4 - 4\varepsilon + \varepsilon^2 < 4 - x < 4 + 4\varepsilon + \varepsilon^2$ or $(2-\varepsilon)^2 < 4 - x < (2+\varepsilon)^2$ or (since the square root function is increasing) $\sqrt{(2-\varepsilon)^2} < \sqrt{4-x} < \sqrt{(2+\varepsilon)^2}$ or $|2-\varepsilon| < \sqrt{4-x} < |2+\varepsilon|$ or (since $0 < \varepsilon < 2$) $2 - \varepsilon < \sqrt{4-x} < 2 + \varepsilon$ or $-\varepsilon < \sqrt{4-x} - 2 < \varepsilon$ or $|\sqrt{4-x} - 2| < \varepsilon$ or $|f(x) - L| < \varepsilon$, as desired.

If $\varepsilon \geq 2$, then choose $\delta = 4(1) - (1)^2 = 3$. Then by the computation above, we have for $0 < |x - 0| < \delta = 3$, we have $|f(x) - L| < 1$ (we repeat the computation above with $\varepsilon = 1$ to establish this). Then we also have $|f(x) - L| < 1 < 2 \leq \varepsilon$, or $|f(x) - L| < \varepsilon$, as desired. $\square$

Example 2.3.6

Example 2.3.6. Prove the Sum Rule, Theorem 2.1(1): If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

$$\lim_{x \to c}(f(x) + g(x)) = \lim_{x \to c}(f(x)) + \lim_{x \to c}(g(x)) = L + M.$$

Proof. First, since $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist, then there is an open interval containing c, say (a_1, b_1), such that f is defined on (a_1, b_1) except possibly at c, and there is an open interval containing c, say (a_2, b_2), such that g is defined on (a_2, b_2) except possibly at c. Define the interval $(a, b) = (a_1, b_1) \cap (a_2, b_2) = (\max\{a_1, a_2\}, \min\{b_1, b_2\})$, and then (a, b) is an open interval containing c where $f + g$ is defined on (a, b), except possibly at c.

Example 2.3.6

Example 2.3.6. Prove the Sum Rule, Theorem 2.1(1): If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

$$\lim_{x \to c}(f(x) + g(x)) = \lim_{x \to c}(f(x)) + \lim_{x \to c}(g(x)) = L + M.$$

Proof. First, since $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist, then there is an open interval containing c, say (a_1, b_1), such that f is defined on (a_1, b_1) except possibly at c, and there is an open interval containing c, say (a_2, b_2), such that g is defined on (a_2, b_2) except possibly at c. Define the interval $(a, b) = (a_1, b_1) \cap (a_2, b_2) = (\max\{a_1, a_2\}, \min\{b_1, b_2\})$, and then (a, b) is an open interval containing c where $f + g$ is defined on (a, b), except possibly at c.

Let $\varepsilon > 0$ be given. Then $\varepsilon/2 > 0$ and since $\lim_{x \to c} f(x) = L$ then by the definition of limit there exists $\delta_1 > 0$ such that $0 < |x - c| < \delta_1$ implies $|f(x) - L| < \varepsilon/2$.

Example 2.3.6

Example 2.3.6. Prove the Sum Rule, Theorem 2.1(1): If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

$$\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c}(f(x)) + \lim_{x\to c}(g(x)) = L + M.$$

Proof. First, since $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then there is an open interval containing c, say (a_1, b_1), such that f is defined on (a_1, b_1) except possibly at c, and there is an open interval containing c, say (a_2, b_2), such that g is defined on (a_2, b_2) except possibly at c. Define the interval $(a, b) = (a_1, b_1) \cap (a_2, b_2) = (\max\{a_1, a_2\}, \min\{b_1, b_2\})$, and then (a, b) is an open interval containing c where $f + g$ is defined on (a, b), except possibly at c.

Let $\varepsilon > 0$ be given. Then $\varepsilon/2 > 0$ and since $\lim_{x\to c} f(x) = L$ then by the definition of limit there exists $\delta_1 > 0$ such that $0 < |x - c| < \delta_1$ implies $|f(x) - L| < \varepsilon/2$.

Example 2.3.6 (continued)

Proof (continued). Similarly, since $\lim_{x\to c} g(x) = M$ then there exists $\delta_2 > 0$ such that $0 < |x - c| < \delta_2$ implies $|g(x) - M| < \varepsilon/2$. We choose $\delta = \min\{\delta_1, \delta_2\}$. Now $0 < |x - c| < \delta$ implies

$$\begin{aligned}
|(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\
&\leq |f(x) - L| + |g(x) - M| \text{ by the Triangle} \\
&\quad \text{Inequality for absolute values} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therefore, by the definition of limit, $\lim_{x\to c}(f(x) + g(x)) = L + M$, as claimed. □

Example 2.3.6 (continued)

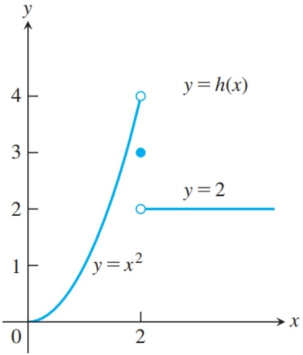
Proof (continued). Similarly, since $\lim_{x \to c} g(x) = M$ then there exists $\delta_2 > 0$ such that $0 < |x - c| < \delta_2$ implies $|g(x) - M| < \varepsilon/2$. We choose $\delta = \min\{\delta_1, \delta_2\}$. Now $0 < |x - c| < \delta$ implies

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \text{ by the Triangle} \\ &\quad \text{Inequality for absolute values} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Therefore, by the definition of limit, $\lim_{x \to c}(f(x) + g(x)) = L + M$, as claimed. □

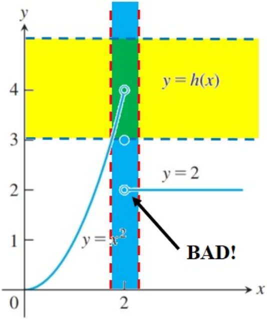
Exercise 2.3.58

Exercise 2.3.58. Use the comment above to show that **(a)** $\lim_{x\to 2} h(x) \neq 4$, **(b)** $\lim_{x\to 2} h(x) \neq 3$, **(c)** $\lim_{x\to 2} h(x) \neq 2$ for the piecewise defined function $h(x) = \begin{cases} x^2, & x < 2 \\ 3, & x = 2 \\ 2, & x > 2. \end{cases}$



Exercise 2.3.58 (continued 1)

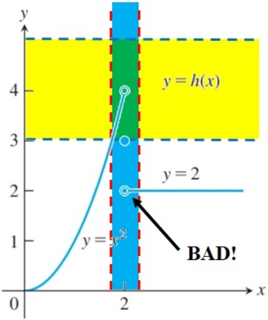
(a) $\lim_{x\to 2} h(x) \neq 4$



Solution. (a) We show that $\varepsilon = 1$ is "bad" in the sense described above. If $L = 4$, then we need the graph of $y = h(x)$ to lie in the yellow band determined by $3 < y < 5$ since $4 - \varepsilon = 4 - 1 = 3$ and $4 + \varepsilon = 4 + 1 = 5$.

Exercise 2.3.58 (continued 1)

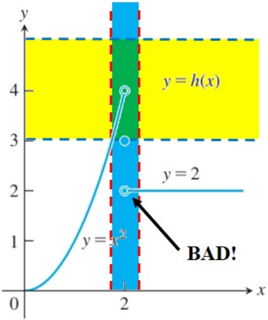
(a) $\lim_{x\to 2} h(x) \neq 4$



Solution. **(a)** We show that $\varepsilon = 1$ is "bad" in the sense described above. If $L = 4$, then we need the graph of $y = h(x)$ to lie in the yellow band determined by $3 < y < 5$ since $4 - \varepsilon = 4 - 1 = 3$ and $4 + \varepsilon = 4 + 1 = 5$. However, no matter how small we make $\delta > 0$, the blue band (of width 2δ and centered at $x = 2$) and yellow band intersect to give the little green box in such a way that there are function values outside of the green box (such points on the graph of $y = h(x)$ are indicated as "**BAD**" above). So the limit is not $L = 4$. □

Exercise 2.3.58 (continued 1)

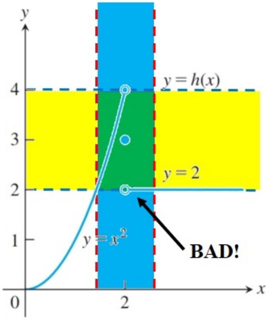
(a) $\lim_{x \to 2} h(x) \neq 4$



Solution. (a) We show that $\varepsilon = 1$ is "bad" in the sense described above. If $L = 4$, then we need the graph of $y = h(x)$ to lie in the yellow band determined by $3 < y < 5$ since $4 - \varepsilon = 4 - 1 = 3$ and $4 + \varepsilon = 4 + 1 = 5$. However, no matter how small we make $\delta > 0$, the blue band (of width 2δ and centered at $x = 2$) and yellow band intersect to give the little green box in such a way that there are function values outside of the green box (such points on the graph of $y = h(x)$ are indicated as "**BAD**" above). So the limit is not $L = 4$. □

Exercise 2.3.58 (continued 2)

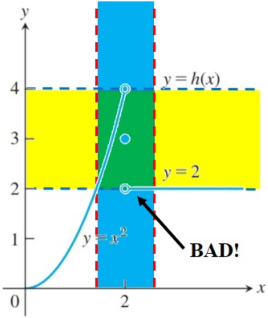
(b) $\lim_{x \to 2} h(x) \neq 3$



Solution. (b) We show that $\varepsilon = 1$ is "bad" in the sense described above. If $L = 3$, then we need the graph of $y = h(x)$ to lie in the yellow band determined by $2 < y < 4$ since $3 - \varepsilon = 3 - 1 = 2$ and $3 + \varepsilon = 3 + 1 = 4$.

Exercise 2.3.58 (continued 2)

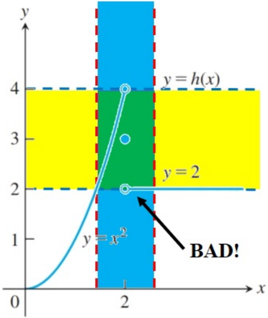
(b) $\lim_{x\to 2} h(x) \neq 3$



Solution. (b) We show that $\varepsilon = 1$ is "bad" in the sense described above. If $L = 3$, then we need the graph of $y = h(x)$ to lie in the yellow band determined by $2 < y < 4$ since $3 - \varepsilon = 3 - 1 = 2$ and $3 + \varepsilon = 3 + 1 = 4$. However, no matter how small we make $\delta > 0$, the blue band (of width 2δ and centered at $x = 2$) and yellow band intersect to give the little green box in such a way that there are function values outside of the green box (such points on the graph of $y = h(x)$ are indicated as "**BAD**" above). So the limit is not $L = 3$. $\square$

Exercise 2.3.58 (continued 2)

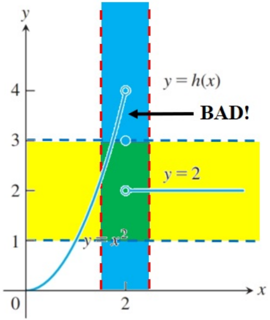
(b) $\lim_{x\to 2} h(x) \neq 3$



Solution. (b) We show that $\varepsilon = 1$ is "bad" in the sense described above. If $L = 3$, then we need the graph of $y = h(x)$ to lie in the yellow band determined by $2 < y < 4$ since $3 - \varepsilon = 3 - 1 = 2$ and $3 + \varepsilon = 3 + 1 = 4$. However, no matter how small we make $\delta > 0$, the blue band (of width 2δ and centered at $x = 2$) and yellow band intersect to give the little green box in such a way that there are function values outside of the green box (such points on the graph of $y = h(x)$ are indicated as "**BAD**" above). So the limit is not $L = 3$. □

Exercise 2.3.58 (continued 3)

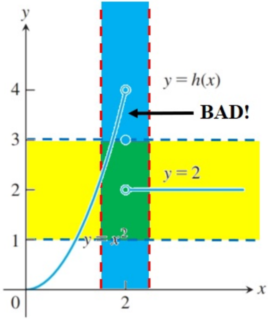
(c) $\lim_{x \to 2} h(x) \neq 2$



Solution. (c) We show that $\varepsilon = 1$ is "bad" in the sense described above. If $L = 2$, then we need the graph of $y = h(x)$ to lie in the yellow band determined by $1 < y < 3$ since $2 - \varepsilon = 2 - 1 = 1$ and $2 + \varepsilon = 2 + 1 = 3$.

Exercise 2.3.58 (continued 3)

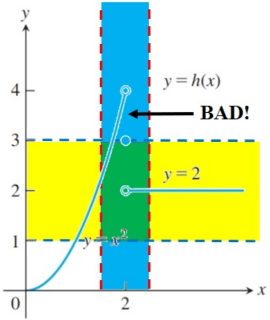
(c) $\lim_{x \to 2} h(x) \neq 2$



Solution. (c) We show that $\varepsilon = 1$ is "bad" in the sense described above. If $L = 2$, then we need the graph of $y = h(x)$ to lie in the yellow band determined by $1 < y < 3$ since $2 - \varepsilon = 2 - 1 = 1$ and $2 + \varepsilon = 2 + 1 = 3$. However, no matter how small we make $\delta > 0$, the blue band (of width 2δ and centered at $x = 2$) and yellow band intersect to give the little green box in such a way that there are function values outside of the green box (such points on the graph of $y = h(x)$ are indicated as "**BAD**" above). So the limit is not $L = 2$. $\square$

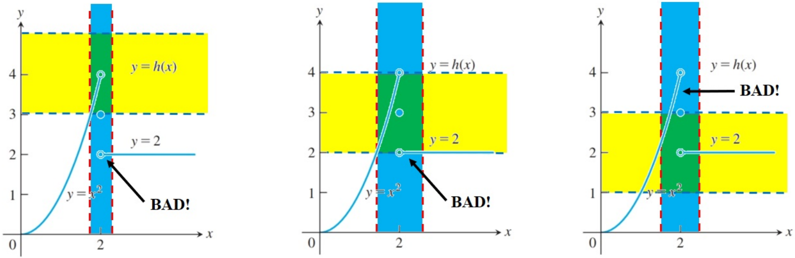
Exercise 2.3.58 (continued 3)

(c) $\lim_{x\to 2} h(x) \neq 2$



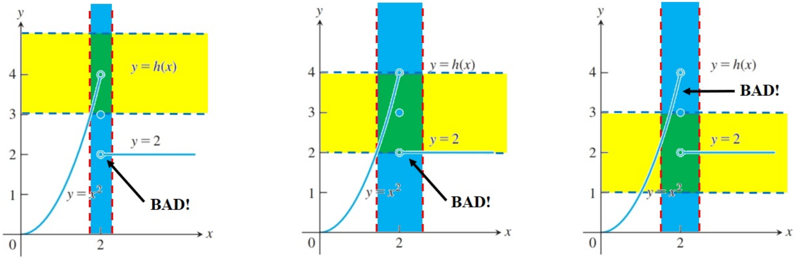
Solution. (c) We show that $\varepsilon = 1$ is "bad" in the sense described above. If $L = 2$, then we need the graph of $y = h(x)$ to lie in the yellow band determined by $1 < y < 3$ since $2 - \varepsilon = 2 - 1 = 1$ and $2 + \varepsilon = 2 + 1 = 3$. However, no matter how small we make $\delta > 0$, the blue band (of width 2δ and centered at $x = 2$) and yellow band intersect to give the little green box in such a way that there are function values outside of the green box (such points on the graph of $y = h(x)$ are indicated as "**BAD**" above). So the limit is not $L = 2$. □

Exercise 2.3.58 (continued 4)



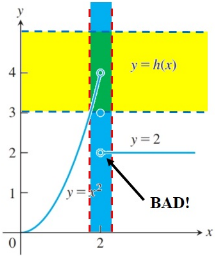
Note. In the first problem, we could have taken ε as big as 2 and it would still have been "bad" because of the behavior of $y = h(x)$ for $x > 2$; the straight-line right-hand part of h lies outside of the yellow band, no matter what $\delta > 0$ is. Any value of $\varepsilon > 2$ would not be bad, since the yellow band would then be wide enough to include all relevant function values in the blue band given above (for example).

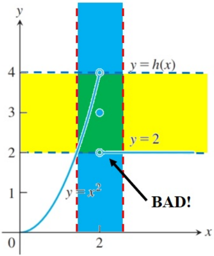
Exercise 2.3.58 (continued 4)

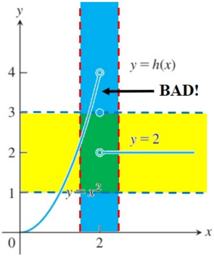


Note. In the first problem, we could have taken ε as big as 2 and it would still have been "bad" because of the behavior of $y = h(x)$ for $x > 2$; the straight-line right-hand part of h lies outside of the yellow band, no matter what $\delta > 0$ is. Any value of $\varepsilon > 2$ would not be bad, since the yellow band would then be wide enough to include all relevant function values in the blue band given above (for example). In the second problem, similar to discussed above, any value of $\varepsilon > 1$ is not bad since the yellow band would then be wide enough to include all relevant function values in the blue band given above (for example).

Exercise 2.3.58 (continued 4)

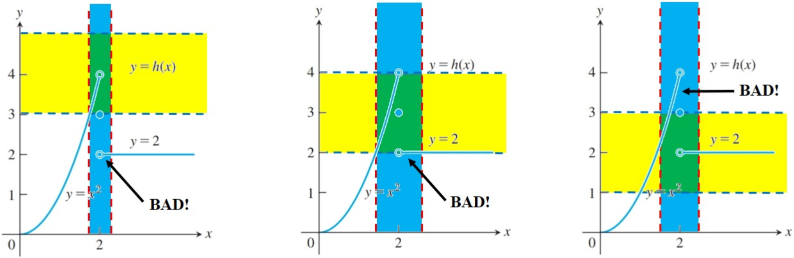






Note. In the first problem, we could have taken ε as big as 2 and it would still have been "bad" because of the behavior of $y = h(x)$ for $x > 2$; the straight-line right-hand part of h lies outside of the yellow band, no matter what $\delta > 0$ is. Any value of $\varepsilon > 2$ would not be bad, since the yellow band would then be wide enough to include all relevant function values in the blue band given above (for example). In the second problem, similar to discussed above, any value of $\varepsilon > 1$ is not bad since the yellow band would then be wide enough to include all relevant function values in the blue band given above (for example). In the third problem, we would need $\varepsilon \geq 2$ in order to include all relevant function values; that is, and $\varepsilon < 2$ if "bad." □

Exercise 2.3.58 (continued 4)



Note. In the first problem, we could have taken ε as big as 2 and it would still have been "bad" because of the behavior of $y = h(x)$ for $x > 2$; the straight-line right-hand part of h lies outside of the yellow band, no matter what $\delta > 0$ is. Any value of $\varepsilon > 2$ would not be bad, since the yellow band would then be wide enough to include all relevant function values in the blue band given above (for example). In the second problem, similar to discussed above, any value of $\varepsilon > 1$ is not bad since the yellow band would then be wide enough to include all relevant function values in the blue band given above (for example). In the third problem, we would need $\varepsilon \geq 2$ in order to include all relevant function values; that is, and $\varepsilon < 2$ if "bad." $\square$