Calculus 1

Chapter 2. Limits and Continuity

2.4. One-Sided Limits—Examples and Proofs

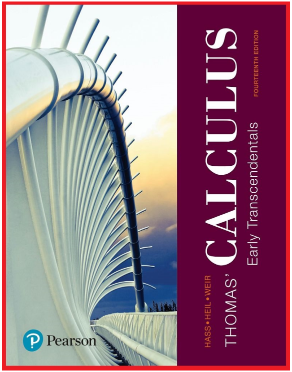
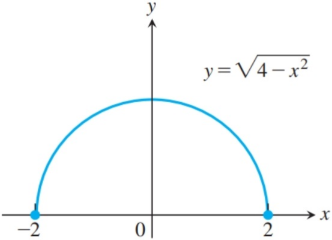


Table of contents

Example 2.4.1

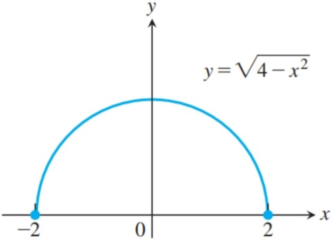
Example 2.4.1. The domain of $f(x) = \sqrt{4 - x^2} = \sqrt{(2-x)(2+x)}$ is $[-2, 2]$; its graph is the semicircle given here: Discuss its one and two sided limits.



Solution. We see from the graph of $y = f(x)$ that, as x approaches -2 from the right (i.e., from the positive side), the graph of the function tries to contain the point $(-2, 0)$. So by an anthropomorphic version of one-sided limits (or the informal definition), $\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$.

Example 2.4.1

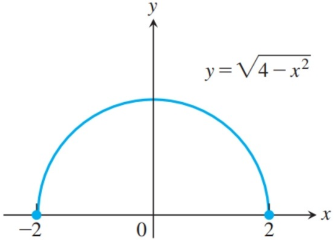
Example 2.4.1. The domain of $f(x) = \sqrt{4 - x^2} = \sqrt{(2-x)(2+x)}$ is $[-2, 2]$; its graph is the semicircle given here: Discuss its one and two sided limits.



Solution. We see from the graph of $y = f(x)$ that, as x approaches -2 from the right (i.e., from the positive side), the graph of the function tries to contain the point $(-2, 0)$. So by an anthropomorphic version of one-sided limits (or the informal definition), $\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$. Similarly, as x approaches $+2$ from the left (i.e., from the negative side), the graph of the function tries to contain the point $(+2, 0)$. So by an anthropomorphic version of one sided limits (or the informal definition), $\lim_{x \to +2^-} \sqrt{4 - x^2} = 0$. Notice that in both cases, the graph succeeds in containing these points (though this is irrelevant to the existence of the limit).

Example 2.4.1

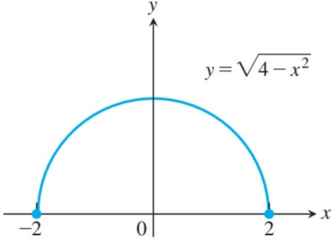
Example 2.4.1. The domain of $f(x) = \sqrt{4 - x^2} = \sqrt{(2 - x)(2 + x)}$ is $[-2, 2]$; its graph is the semicircle given here: Discuss its one and two sided limits.



Solution. We see from the graph of $y = f(x)$ that, as x approaches -2 from the right (i.e., from the positive side), the graph of the function tries to contain the point $(-2, 0)$. So by an anthropomorphic version of one-sided limits (or the informal definition), $\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$. Similarly, as x approaches $+2$ from the left (i.e., from the negative side), the graph of the function tries to contain the point $(+2, 0)$. So by an anthropomorphic version of one sided limits (or the informal definition), $\lim_{x \to +2^-} \sqrt{4 - x^2} = 0$. Notice that in both cases, the graph succeeds in containing these points (though this is irrelevant to the existence of the limit).

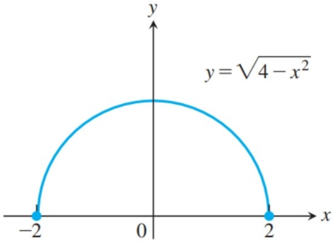
Example 2.4.1 (continued 1)

Solution (continued). For any other c with $-2 < c < 2$, we see that the function tries to pass though the points $(c, f(c)) = (c, \sqrt{4-(c)^2})$ (and succeeds) so that by Dr. Bob's Anthropomorphic Definition of Limit (or the informal definition or the formal definition), the (two-sided) limit exists for each such c.



Notice that the two-sided limit (or simply "limit") at $c = \pm 2$ does not exist. This is because there is not an open interval containing $c = \pm 2$ on which f is defined, except possibly at $c = \pm 2$; notice that $f(x)$ is not defined for $x < -2$ and $f(x)$ is not defined for $x > +2$.

Example 2.4.1 (continued 1)



Solution (continued). For any other c with $-2 < c < 2$, we see that the function tries to pass though the points $(c, f(c)) = (c, \sqrt{4 - (c)^2})$ (and succeeds) so that by Dr. Bob's Anthropomorphic Definition of Limit (or the informal definition or the formal definition), the (two-sided) limit exists for each such c.

Notice that the two-sided limit (or simply "limit") at $c = \pm 2$ does not exist. This is because there is not an open interval containing $c = \pm 2$ on which f is defined, except possibly at $c = \pm 2$; notice that $f(x)$ is not defined for $x < -2$ and $f(x)$ is not defined for $x > +2$.

Example 2.4.1 (continued 2)

Note. As just argued, neither $\lim_{x\to-2^-} f(x)$ nor $\lim_{x\to+2^+} f(x)$ exist. This shows that the evaluation of limits is more complicated than substituting in a value when there is no division by 0. *If* we substitute $x = \pm 2$ into $f(x) = \sqrt{4-x^2}$ then we simply get $f(\pm 2) = 0$ (and 0 is the value of the one-sided limits which exist); but these are not the values of the two-sided limits since these do not exist! The problem that arises in the two sided limits is the square roots of negatives. Notice that when x is "close to" -2 then x could be less than -2 yielding square roots of negatives for $f(x) = \sqrt{4-x^2}$. Similarly when x is "close to" $+2$ then x could be greater than $+2$ yielding square roots of negatives for $f(x) = \sqrt{4-x^2}$. *This* is why the two-sided limits don't exist. □

Example 2.4.1 (continued 2)

Note. As just argued, neither $\lim_{x\to -2^-} f(x)$ nor $\lim_{x\to +2^+} f(x)$ exist. This shows that the evaluation of limits is more complicated than substituting in a value when there is no division by 0. *If* we substitute $x = \pm 2$ into $f(x) = \sqrt{4 - x^2}$ then we simply get $f(\pm 2) = 0$ (and 0 is the value of the one-sided limits which exist); but these are not the values of the two-sided limits since these do not exist! The problem that arises in the two sided limits is the square roots of negatives. Notice that when x is "close to" -2 then x could be less than -2 yielding square roots of negatives for $f(x) = \sqrt{4 - x^2}$. Similarly when x is "close to" $+2$ then x could be greater than $+2$ yielding square roots of negatives for $f(x) = \sqrt{4 - x^2}$. *This* is why the two-sided limits don't exist. $\square$

Exercise 2.4.10

Exercise 2.4.10. Consider

$$f(x) = \begin{cases} x, & -1 \leq x < 0 \text{ or } 0 < x \leq 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$$

Graph $y = f(x)$.

(a) What are the domain and range of f?

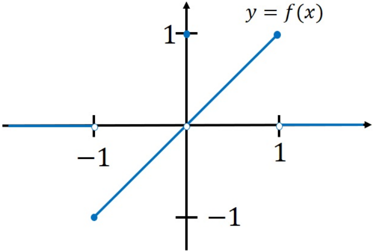
(b) At what points c, if any, does $\lim_{x \to c} f(x)$ exist?

(c) At what points does the left-hand limit exist but not the right-hand limit?

(d) At what points does the right-hand limit exist but not the left-hand limit?

Exercise 2.4.10 (continued 1)

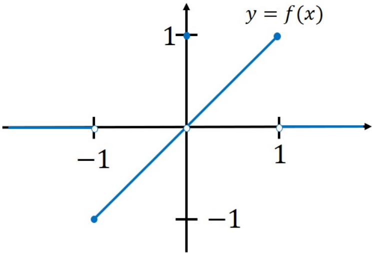
$$f(x) = \begin{cases} x, & -1 \leq x < 0 \text{ or } 0 < x \leq 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$$



Solution. (a) We see from the graph that the domain of f is all of $\mathbb{R}$ and the range of f is $[-1, 1]$. □

Exercise 2.4.10 (continued 1)

$$f(x) = \begin{cases} x, & -1 \le x < 0 \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$$

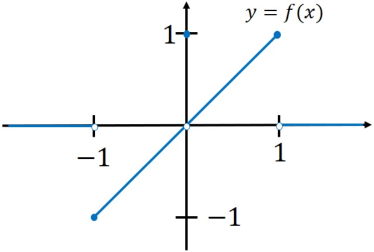


Solution. **(a)** We see from the graph that the $\boxed{\text{domain of } f \text{ is all of } \mathbb{R}}$ and the $\boxed{\text{range of } f \text{ is } [-1, 1]}$. □

(b) We use Dr. Bob's Anthropomorphic Definition of Limit to explore $\lim_{x \to c} f(x)$. For $c < -1$ and $c > 1$ the graph of f tries (and succeeds) to pass through the point $(c, 0)$ so for these c values the limit exists. For $-1 < c < 1$ the graph of f tries to pass through the point (c, c) (and succeeds, except when $c = 0$) so for these c values the limit exists.

Exercise 2.4.10 (continued 1)

$$f(x) = \begin{cases} x, & -1 \leq x < 0 \text{ or } 0 < x \leq 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$$

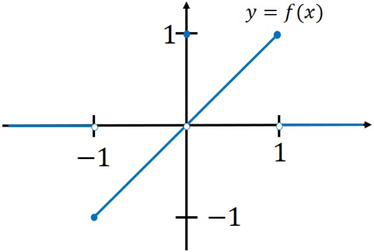


Solution. **(a)** We see from the graph that the domain of f is all of $\mathbb{R}$ and the range of f is $[-1, 1]$. □

(b) We use Dr. Bob's Anthropomorphic Definition of Limit to explore $\lim_{x \to c} f(x)$. For $c < -1$ and $c > 1$ the graph of f tries (and succeeds) to pass through the point $(c, 0)$ so for these c values the limit exists. For $-1 < c < 1$ the graph of f tries to pass through the point (c, c) (and succeeds, except when $c = 0$) so for these c values the limit exists.

Exercise 2.4.10 (continued 2)

$$f(x) = \begin{cases} x, & -1 \leq x < 0 \text{ or } 0 < x \leq 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$$

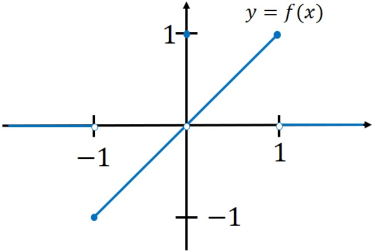


Solution (continued). For $c = \pm 1$ there is no single point through which that the graph of f tries to pass for x near c, so for these c values the limit does not exist. So $\lim_{x \to c} f(x)$ exists for $\boxed{c \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)}$. □

(c,d) Notice that for all c, the graph of f tries to pass through some point as x approaches c from the left (by a one-sided version of Dr. Bob's Anthropomorphic Definition of Limit, or by the Informal Definition of Left-Hand Limits). So $\lim_{x \to c^-} f(x)$ exists for all c.

Exercise 2.4.10 (continued 2)

$$f(x) = \begin{cases} x, & -1 \leq x < 0 \text{ or } 0 < x \leq 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$$

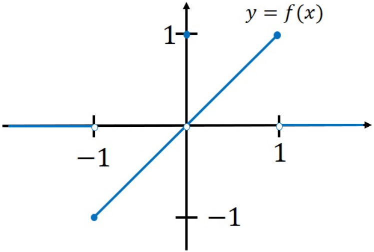


Solution (continued). For $c = \pm 1$ there is no single point through which that the graph of f tries to pass for x near c, so for these c values the limit does not exist. So $\lim_{x \to c} f(x)$ exists for $\boxed{c \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)}$. □

(c,d) Notice that for all c, the graph of f tries to pass through some point as x approaches c from the left (by a one-sided version of Dr. Bob's Anthropomorphic Definition of Limit, or by the Informal Definition of Left-Hand Limits). So $\lim_{x \to c^-} f(x)$ exists for all c.

Exercise 2.4.10 (continued 3)

$$f(x) = \begin{cases} x, & -1 \le x < 0 \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$$

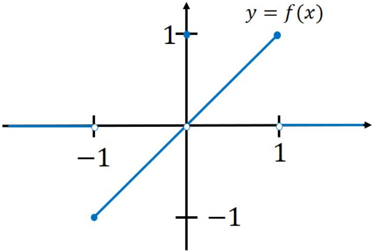


Solution (c,d) (continued). Similarly, for all c, the graph of f tries to pass through some point as x approaches c from the right (by a one-sided version of Dr. Bob's Anthropomorphic Definition of Limit, or by the Informal Definition of Right-Hand Limits). So $\lim_{x \to c^+} f(x)$ exists for all c. Hence, there are

no points c where just one of the one-sided limits exist. □

Exercise 2.4.10 (continued 3)

$$f(x) = \begin{cases} x, & -1 \le x < 0 \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$$



Solution (c,d) (continued). Similarly, for all c, the graph of f tries to pass through some point as x approaches c from the right (by a one-sided version of Dr. Bob's Anthropomorphic Definition of Limit, or by the Informal Definition of Right-Hand Limits). So $\lim_{x \to c^+} f(x)$ exists for all c. Hence, there are

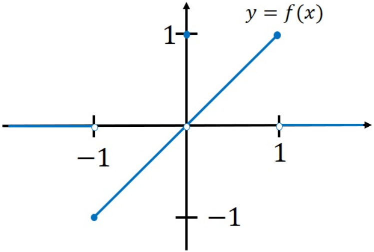
no points c where just one of the one-sided limits exist.

□

Note. The left-hand and right-hand limits are the same at all points c, except for $c = \pm 1$. □

Exercise 2.4.10 (continued 3)

$$f(x) = \begin{cases} x, & -1 \le x < 0 \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$$



Solution (c,d) (continued). Similarly, for all c, the graph of f tries to pass through some point as x approaches c from the right (by a one-sided version of Dr. Bob's Anthropomorphic Definition of Limit, or by the Informal Definition of Right-Hand Limits). So $\lim_{x \to c^+} f(x)$ exists for all c. Hence, there are

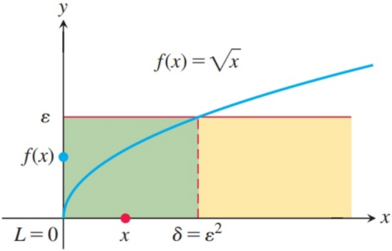
$\boxed{\text{no points } c \text{ where just one of the one-sided limits exist.}}$ □

Note. The left-hand and right-hand limits are the same at all points c, except for $c = \pm 1$. □

Example 2.4.3

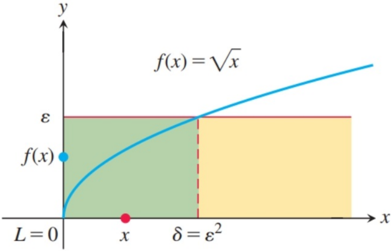
Example 2.4.3. Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Proof. First, we need $f(x) = \sqrt{x}$ defined on an open interval of the form $(c, b) = (0, b)$. The is the case since the domain of f is $[0, \infty)$ so that we have f defined on (say) $(c, 1) = (0, 1)$. Now let $\varepsilon > 0$.



Example 2.4.3

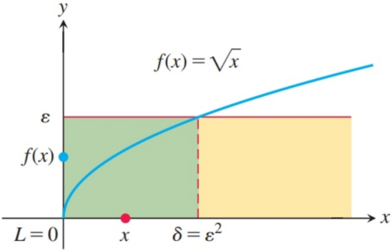
Example 2.4.3. Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.



Proof. First, we need $f(x) = \sqrt{x}$ defined on an open interval of the form $(c, b) = (0, b)$. The is the case since the domain of f is $[0, \infty)$ so that we have f defined on (say) $(c, 1) = (0, 1)$. Now let $\varepsilon > 0$. [Not part of the proof: We see from the graph above that in order to get $f(x)$ within a distance of ε of $L = 0$, we need to have x in the interval $[0, \varepsilon^2)$.] Choose $\delta = \varepsilon^2$.

Example 2.4.3

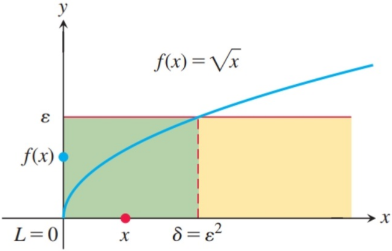
Example 2.4.3. Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.



Proof. First, we need $f(x) = \sqrt{x}$ defined on an open interval of the form $(c, b) = (0, b)$. The is the case since the domain of f is $[0, \infty)$ so that we have f defined on (say) $(c, 1) = (0, 1)$. Now let $\varepsilon > 0$. [Not part of the proof: We see from the graph above that in order to get $f(x)$ within a distance of ε of $L = 0$, we need to have x in the interval $[0, \varepsilon^2)$.] Choose $\delta = \varepsilon^2$. If $c < x < c + \delta$, or equivalently $0 < x < 0 + \delta = \varepsilon^2$, then (since the square root function is increasing for nonnegative inputs) $\sqrt{x} < \sqrt{\varepsilon^2} = |\varepsilon| = \varepsilon$, or equivalently $|\sqrt{x} - 0| = |f(x) - L| < \varepsilon$ where $L = 0$. Therefore, by the Formal Definitions of One-Sided Limits, $\lim_{x \to 0^+} f(x) = L$ or $\lim_{x \to 0^+} \sqrt{x} = 0$. □

Example 2.4.3

Example 2.4.3. Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.



Proof. First, we need $f(x) = \sqrt{x}$ defined on an open interval of the form $(c, b) = (0, b)$. The is the case since the domain of f is $[0, \infty)$ so that we have f defined on (say) $(c, 1) = (0, 1)$. Now let $\varepsilon > 0$. [Not part of the proof: We see from the graph above that in order to get $f(x)$ within a distance of ε of $L = 0$, we need to have x in the interval $[0, \varepsilon^2)$.] Choose $\delta = \varepsilon^2$. If $c < x < c + \delta$, or equivalently $0 < x < 0 + \delta = \varepsilon^2$, then (since the square root function is increasing for nonnegative inputs) $\sqrt{x} < \sqrt{\varepsilon^2} = |\varepsilon| = \varepsilon$, or equivalently $|\sqrt{x} - 0| = |f(x) - L| < \varepsilon$ where $L = 0$. Therefore, by the Formal Definitions of One-Sided Limits, $\lim_{x \to 0^+} f(x) = L$ or $\lim_{x \to 0^+} \sqrt{x} = 0$. □

Exercise 2.4.50

Exercise 2.4.50. Suppose that f is an even function of x. Does knowing that $\lim_{x\to 2^-} f(x) = 7$ tell you anything about either $\lim_{x\to -2^-} f(x)$ or $\lim_{x\to -2^+} f(x)$? Give reasons for your answer.

Solution. Recall that an even function satisfies $f(-x) = f(x)$. When considering $x \to 2^-$, we have x in some interval of the form $(2-\delta, 2)$. That is, we consider $2-\delta < x < 2$. Then $-(2-\delta) > -x > -2$ or $-2 < -x < -2+\delta$.

Exercise 2.4.50

Exercise 2.4.50. Suppose that f is an even function of x. Does knowing that $\lim_{x\to 2^-} f(x) = 7$ tell you anything about either $\lim_{x\to -2^-} f(x)$ or $\lim_{x\to -2^+} f(x)$? Give reasons for your answer.

Solution. Recall that an even function satisfies $f(-x) = f(x)$. When considering $x \to 2^-$, we have x in some interval of the form $(2-\delta, 2)$. That is, we consider $2-\delta < x < 2$. Then $-(2-\delta) > -x > -2$ or $-2 < -x < -2+\delta$. Since $f(x) = f(-x)$, the behavior of $f(x)$ for $2-\delta < x < 2$ is the same as the behavior of $f(-x) = f(x)$ for $-2 < -x < -2+\delta$. So if $|f(x) - 7| < \varepsilon$ for $2-\delta < x < 2$, then $|f(-x) - 7| < \varepsilon$ for $-2 < -x < -2+\delta$; or (substituting x for $-x$ in the last claim) $|f(x) - 7| < \varepsilon$ for $-2 < x < -2+\delta$. So we must have $\boxed{\lim_{x\to -2^+} f(x) = 7}$.

Exercise 2.4.50

Exercise 2.4.50. Suppose that f is an even function of x. Does knowing that $\lim_{x\to 2^-} f(x) = 7$ tell you anything about either $\lim_{x\to -2^-} f(x)$ or $\lim_{x\to -2^+} f(x)$? Give reasons for your answer.

Solution. Recall that an even function satisfies $f(-x) = f(x)$. When considering $x \to 2^-$, we have x in some interval of the form $(2-\delta, 2)$. That is, we consider $2-\delta < x < 2$. Then $-(2-\delta) > -x > -2$ or $-2 < -x < -2+\delta$. Since $f(x) = f(-x)$, the behavior of $f(x)$ for $2-\delta < x < 2$ is the same as the behavior of $f(-x) = f(x)$ for $-2 < -x < -2+\delta$. So if $|f(x) - 7| < \varepsilon$ for $2-\delta < x < 2$, then $|f(-x) - 7| < \varepsilon$ for $-2 < -x < -2+\delta$; or (substituting x for $-x$ in the last claim) $|f(x) - 7| < \varepsilon$ for $-2 < x < -2+\delta$. So we must have $\boxed{\lim_{x\to -2^+} f(x) = 7}$.

$\boxed{\text{We know nothing about } \lim_{x\to -2^-} f(x)}$; if we knew something about $\lim_{x\to 2^+} f(x)$ then we could use that information to deduce the value of $\lim_{x\to -2^-} f(x)$ using the "evenness" of f, as above. □

Exercise 2.4.50

Exercise 2.4.50. Suppose that f is an even function of x. Does knowing that $\lim_{x\to 2^-} f(x) = 7$ tell you anything about either $\lim_{x\to -2^-} f(x)$ or $\lim_{x\to -2^+} f(x)$? Give reasons for your answer.

Solution. Recall that an even function satisfies $f(-x) = f(x)$. When considering $x \to 2^-$, we have x in some interval of the form $(2-\delta, 2)$. That is, we consider $2-\delta < x < 2$. Then $-(2-\delta) > -x > -2$ or $-2 < -x < -2+\delta$. Since $f(x) = f(-x)$, the behavior of $f(x)$ for $2-\delta < x < 2$ is the same as the behavior of $f(-x) = f(x)$ for $-2 < -x < -2+\delta$. So if $|f(x) - 7| < \varepsilon$ for $2-\delta < x < 2$, then $|f(-x) - 7| < \varepsilon$ for $-2 < -x < -2+\delta$; or (substituting x for $-x$ in the last claim) $|f(x) - 7| < \varepsilon$ for $-2 < x < -2+\delta$. So we must have $\boxed{\lim_{x\to -2^+} f(x) = 7}$.

$\boxed{\text{We know nothing about } \lim_{x\to -2^-} f(x)}$; if we knew something about $\lim_{x\to 2^+} f(x)$ then we could use that information to deduce the value of $\lim_{x\to -2^-} f(x)$ using the "evenness" of f, as above. □

Theorem 2.7

Theorem 2.7. Limit of the Ratio $(\sin\theta)/\theta$ as $\theta \to 0$.
For θ in radians, $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$

Proof. Suppose first that θ is positive and less than $\pi/2$. Consider the picture:

Theorem 2.7

Theorem 2.7. Limit of the Ratio $(\sin\theta)/\theta$ **as** $\theta \to 0$**.**

For θ in radians, $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$

Proof. Suppose first that θ is positive and less than $\pi/2$. Consider the picture:

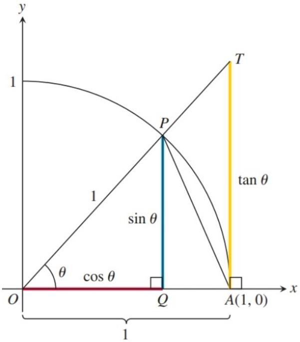


Figure 2.33

Theorem 2.7

Theorem 2.7. Limit of the Ratio $(\sin\theta)/\theta$ as $\theta \to 0$.

For θ in radians, $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

Proof. Suppose first that θ is positive and less than $\pi/2$. Consider the picture:

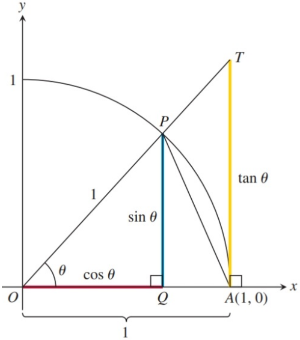


Figure 2.33

Notice that

Area $\triangle OAP <$ area sector $OAP <$ area $\triangle OAT$.

We can express these areas in terms of θ as follows: Area $\triangle OAP = \frac{1}{2}$ base $\times$ height

$$= \tfrac{1}{2}(1)(\sin\theta) = \tfrac{1}{2}\sin\theta,$$

$$\text{Area sector } OAP = \tfrac{1}{2}r^2\theta = \tfrac{1}{2}(1)^2\theta = \tfrac{\theta}{2},$$

$$\text{Area } \triangle OAT = \tfrac{1}{2} \text{ base } \times \text{ height}$$

$$= \tfrac{1}{2}(1)(\tan\theta) = \tfrac{1}{2}\tan\theta.$$

Thus, $\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$.

Theorem 2.7

Theorem 2.7. Limit of the Ratio $(\sin\theta)/\theta$ as $\theta \to 0$.
For θ in radians, $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

Proof. Suppose first that θ is positive and less than $\pi/2$. Consider the picture:

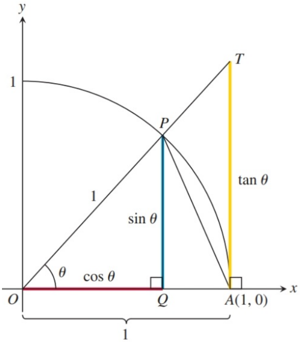


Figure 2.33

Notice that

Area $\triangle OAP <$ area sector $OAP <$ area $\triangle OAT$.

We can express these areas in terms of θ as follows: Area $\triangle OAP = \frac{1}{2}$ base $\times$ height

$$= \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta,$$

$$\text{Area sector } OAP = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2},$$

$$\text{Area } \triangle OAT = \frac{1}{2} \text{ base } \times \text{ height}$$

$$= \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta.$$

Thus, $\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$.

Theorem 2.7 (continued)

Proof (continued). Thus, $\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$. Dividing all three terms in this inequality by the positive number $(1/2)\sin\theta$ gives: $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$. Taking reciprocals reverses the inequalities: $\cos\theta < \frac{\sin\theta}{\theta} < 1$. Since $\lim_{\theta\to 0^+}\cos\theta = 1$ by Example 2.2.11(b), the Sandwich Theorem (applied to the one-sided limit) gives $\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$.

Theorem 2.7 (continued)

Proof (continued). Thus, $\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$. Dividing all three terms in this inequality by the positive number $(1/2)\sin\theta$ gives: $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$. Taking reciprocals reverses the inequalities: $\cos\theta < \frac{\sin\theta}{\theta} < 1$. Since $\lim_{\theta\to 0^+}\cos\theta = 1$ by Example 2.2.11(b), the Sandwich Theorem (applied to the one-sided limit) gives $\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$.

Since $\sin\theta$ and θ are both odd functions, $f(\theta) = \frac{\sin\theta}{\theta}$ is an even function and hence $\frac{\sin(-\theta)}{-\theta} = \frac{\sin\theta}{\theta}$.

Theorem 2.7 (continued)

Proof (continued). Thus, $\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$. Dividing all three terms in this inequality by the positive number $(1/2)\sin\theta$ gives: $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$. Taking reciprocals reverses the inequalities: $\cos\theta < \frac{\sin\theta}{\theta} < 1$. Since $\lim_{\theta\to 0^+}\cos\theta = 1$ by Example 2.2.11(b), the Sandwich Theorem (applied to the one-sided limit) gives $\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$.

Since $\sin\theta$ and θ are both odd functions, $f(\theta) = \frac{\sin\theta}{\theta}$ is an even function and hence $\frac{\sin(-\theta)}{-\theta} = \frac{\sin\theta}{\theta}$. Therefore $\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}$, so $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ by Theorem 2.6 (Relation Between One-Sided and Two-Sided Limits). □

Theorem 2.7 (continued)

Proof (continued). Thus, $\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$. Dividing all three terms in this inequality by the positive number $(1/2)\sin\theta$ gives: $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$. Taking reciprocals reverses the inequalities: $\cos\theta < \frac{\sin\theta}{\theta} < 1$. Since $\lim_{\theta\to 0^+}\cos\theta = 1$ by Example 2.2.11(b), the Sandwich Theorem (applied to the one-sided limit) gives $\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$. Since $\sin\theta$ and θ are both odd functions, $f(\theta) = \frac{\sin\theta}{\theta}$ is an even function and hence $\frac{\sin(-\theta)}{-\theta} = \frac{\sin\theta}{\theta}$. Therefore $\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}$, so $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ by Theorem 2.6 (Relation Between One-Sided and Two-Sided Limits). □

Example 2.4.5(a)

Example 2.4.5(a) Show that $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$.

Solution. We multiply by $\frac{\cos h + 1}{\cos h + 1}$ to get

$$\begin{aligned}
\lim_{h \to 0} \frac{\cos h - 1}{h} &= \lim_{h \to 0} \frac{\cos h - 1}{h} \left(\frac{\cos h + 1}{\cos h + 1} \right) \\
&= \lim_{h \to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} = \lim_{h \to 0} \frac{\sin^2 h}{h(\cos h + 1)} \\
&= \lim_{h \to 0} \frac{\sin h}{h} \frac{\sin h}{\cos h + 1} \\
&= \lim_{h \to 0} \frac{\sin h}{h} \lim_{h \to 0} \frac{\sin h}{\cos h + 1} \text{ by Theorem 2.1(4)} \\
&\quad \text{(Product Rule)}
\end{aligned}$$

Example 2.4.5(a)

Example 2.4.5(a) Show that $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$.

Solution. We multiply by $\frac{\cos h + 1}{\cos h + 1}$ to get

$$\begin{aligned}
\lim_{h\to 0} \frac{\cos h - 1}{h} &= \lim_{h\to 0} \frac{\cos h - 1}{h} \left(\frac{\cos h + 1}{\cos h + 1}\right) \\
&= \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} = \lim_{h\to 0} \frac{\sin^2 h}{h(\cos h + 1)} \\
&= \lim_{h\to 0} \frac{\sin h}{h} \frac{\sin h}{\cos h + 1} \\
&= \lim_{h\to 0} \frac{\sin h}{h} \lim_{h\to 0} \frac{\sin h}{\cos h + 1} \text{ by Theorem 2.1(4)} \\
&\quad \text{(Product Rule)}
\end{aligned}$$

Example 2.4.5(a) (continued)

Example 2.4.5(a) Show that $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$

Solution (continued).

$$\begin{aligned}
\lim_{h\to 0} \frac{\cos h - 1}{h} &= \lim_{h\to 0} \frac{\sin h}{h} \lim_{h\to 0} \frac{\sin h}{\cos h + 1} \\
&= (1) \lim_{h\to 0} \frac{\sin h}{\cos h + 1} \text{ by Theorem 2.7} \\
&= \frac{\lim_{h\to 0} \sin h}{\lim_{h\to 0} \cos h + 1} \text{ by Theorem 2.1(5) (Quotient Rule)} \\
&= \frac{\sin 0}{\cos 0 + 1} = \frac{0}{1+1} = 0 \text{ by Example 2.2.11.}
\end{aligned}$$

□

Exercise 2.4.28

Exercise 2.4.28. Evaluate $\lim_{t \to 0} \frac{2t}{\tan t}$.

Solution. We have

$$\begin{aligned} \lim_{t \to 0} \frac{2t}{\tan t} &= \lim_{t \to 0} \frac{2t}{(\sin t)/(\cos t)} \\ &= 2 \lim_{t \to 0} \frac{t \cos t}{\sin t} \text{ by Theorem 2.1(3) Constant Multiple Rule} \\ &= 2 \lim_{t \to 0} \frac{t}{\sin t} \lim_{t \to 0} \cos t \text{ by Theorem 2.1(4) (Product Rule)} \\ &= 2 \lim_{t \to 0} \frac{1}{(\sin t)/t} \lim_{t \to 0} \cos t \\ &= 2 \frac{\lim_{t \to 0} 1}{\lim_{t \to 0} (\sin t)/t} \lim_{t \to 0} \cos t \text{ by Theorem 2.1(5)} \\ &\quad \text{(Quotient Rule)} \end{aligned}$$

Exercise 2.4.28

Exercise 2.4.28. Evaluate $\displaystyle\lim_{t\to 0}\frac{2t}{\tan t}$.

Solution. We have

$$\begin{aligned}
\lim_{t\to 0}\frac{2t}{\tan t} &= \lim_{t\to 0}\frac{2t}{(\sin t)/(\cos t)}\\
&= 2\lim_{t\to 0}\frac{t\cos t}{\sin t} \text{ by Theorem 2.1(3) Constant Multiple Rule}\\
&= 2\lim_{t\to 0}\frac{t}{\sin t}\lim_{t\to 0}\cos t \text{ by Theorem 2.1(4) (Product Rule)}\\
&= 2\lim_{t\to 0}\frac{1}{(\sin t)/t}\lim_{t\to 0}\cos t\\
&= 2\frac{\lim_{t\to 0}1}{\lim_{t\to 0}(\sin t)/t}\lim_{t\to 0}\cos t \text{ by Theorem 2.1(5)}\\
&\quad \text{(Quotient Rule)}
\end{aligned}$$

Exercise 2.4.28 (continued)

Exercise 2.4.28. Evaluate $\lim_{t\to 0} \frac{2t}{\tan t}$.

Solution (continued).

$$\begin{aligned}
\lim_{t\to 0} \frac{2t}{\tan t} &= 2\frac{\lim_{t\to 0} 1}{\lim_{t\to 0}(\sin t)/t} \lim_{t\to 0} \cos t \\
&= 2\frac{(1)}{(1)} \cos 0 \text{ by Example 2.3.3(b), Theorem 2.7,} \\
&\quad \text{and Example 2.2.11(a)(b)} \\
&= 2.
\end{aligned}$$

□

Exercise 2.4.52

Exercise 2.4.52. Given $\varepsilon > 0$, find $\delta > 0$ where $I = (4 - \delta, 4)$ is such that if x lies in I, then $\sqrt{4 - x} < \varepsilon$. What limit is being verified and what is its value?

Solution. We let $c = 4$ and $f(x) = \sqrt{4 - x}$. We want $x \in (c - \delta, c) = (4 - \delta, 4)$ to imply $|f(x) - L| = |\sqrt{4 - x} - 0| = \sqrt{4 - x} < \varepsilon$. So we take $L = 0$.

Exercise 2.4.52

Exercise 2.4.52. Given $\varepsilon > 0$, find $\delta > 0$ where $I = (4 - \delta, 4)$ is such that if x lies in I, then $\sqrt{4 - x} < \varepsilon$. What limit is being verified and what is its value?

Solution. We let $c = 4$ and $f(x) = \sqrt{4 - x}$. We want $x \in (c - \delta, c) = (4 - \delta, 4)$ to imply $|f(x) - L| = |\sqrt{4 - x} - 0| = \sqrt{4 - x} < \varepsilon$. So we take $L = 0$. Now $x \in (4 - \delta, 4)$ means $4 - \delta < x < 4$ or $-\delta < x - 4 < 0$ or $0 < 4 - x < \delta$. The implies $\sqrt{0} < \sqrt{4 - x} < \sqrt{\delta}$ since the square root function is an increasing function.

Exercise 2.4.52

Exercise 2.4.52. Given $\varepsilon > 0$, find $\delta > 0$ where $I = (4-\delta, 4)$ is such that if x lies in I, then $\sqrt{4-x} < \varepsilon$. What limit is being verified and what is its value?

Solution. We let $c = 4$ and $f(x) = \sqrt{4-x}$. We want $x \in (c-\delta, c) = (4-\delta, 4)$ to imply $|f(x) - L| = |\sqrt{4-x} - 0| = \sqrt{4-x} < \varepsilon$. So we take $L = 0$. Now $x \in (4-\delta, 4)$ means $4-\delta < x < 4$ or $-\delta < x-4 < 0$ or $0 < 4-x < \delta$. The implies $\sqrt{0} < \sqrt{4-x} < \sqrt{\delta}$ since the square root function is an increasing function. Therefore we need $\sqrt{\delta} \le \varepsilon$, or $\delta \le \varepsilon^2$. In order to keep $I = (4-\delta, 4)$ a subset of the domain of f, we take $\boxed{\delta = \min\{\varepsilon^2, 4\}}$.

Exercise 2.4.52

Exercise 2.4.52. Given $\varepsilon > 0$, find $\delta > 0$ where $I = (4-\delta, 4)$ is such that if x lies in I, then $\sqrt{4-x} < \varepsilon$. What limit is being verified and what is its value?

Solution. We let $c = 4$ and $f(x) = \sqrt{4-x}$. We want $x \in (c-\delta, c) = (4-\delta, 4)$ to imply $|f(x) - L| = |\sqrt{4-x} - 0| = \sqrt{4-x} < \varepsilon$. So we take $L = 0$. Now $x \in (4-\delta, 4)$ means $4-\delta < x < 4$ or $-\delta < x-4 < 0$ or $0 < 4-x < \delta$. The implies $\sqrt{0} < \sqrt{4-x} < \sqrt{\delta}$ since the square root function is an increasing function. Therefore we need $\sqrt{\delta} \le \varepsilon$, or $\delta \le \varepsilon^2$. In order to keep $I = (4-\delta, 4)$ a subset of the domain of f, we take $\boxed{\delta = \min\{\varepsilon^2, 4\}}$.

We have $f(x) = \sqrt{4-x}$, $c = 4$, and $L = 0$. Since we consider x such that $4-\delta < x < 4$, then we are considering a limit from the negative side as x approaches $c = 4$. So the limit being verified is $\boxed{\lim_{x\to 4^-} \sqrt{4-x} = 0}$. $\square$

Exercise 2.4.52

Exercise 2.4.52. Given $\varepsilon > 0$, find $\delta > 0$ where $I = (4 - \delta, 4)$ is such that if x lies in I, then $\sqrt{4-x} < \varepsilon$. What limit is being verified and what is its value?

Solution. We let $c = 4$ and $f(x) = \sqrt{4-x}$. We want $x \in (c - \delta, c) = (4 - \delta, 4)$ to imply $|f(x) - L| = |\sqrt{4-x} - 0| = \sqrt{4-x} < \varepsilon$. So we take $L = 0$. Now $x \in (4 - \delta, 4)$ means $4 - \delta < x < 4$ or $-\delta < x - 4 < 0$ or $0 < 4 - x < \delta$. The implies $\sqrt{0} < \sqrt{4-x} < \sqrt{\delta}$ since the square root function is an increasing function. Therefore we need $\sqrt{\delta} \leq \varepsilon$, or $\delta \leq \varepsilon^2$. In order to keep $I = (4 - \delta, 4)$ a subset of the domain of f, we take $\boxed{\delta = \min\{\varepsilon^2, 4\}}$. We have $f(x) = \sqrt{4-x}$, $c = 4$, and $L = 0$. Since we consider x such that $4 - \delta < x < 4$, then we are considering a limit from the negative side as x approaches $c = 4$. So the limit being verified is $\boxed{\lim_{x \to 4^-} \sqrt{4-x} = 0}$. $\square$