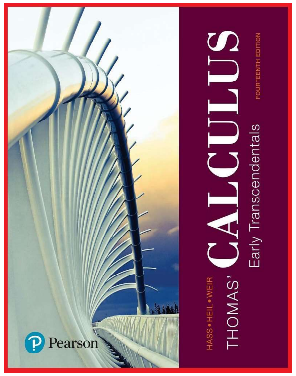
Calculus 1

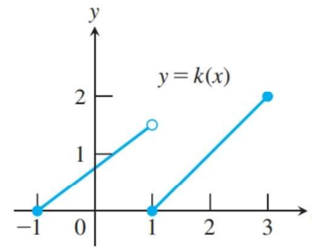
Chapter 2. Limits and Continuity

2.5. Continuity—Examples and Proofs



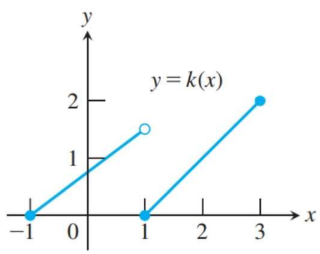
Exercise 2.5.4

Exercise 2.5.4. State whether the function $y = k(x)$ is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?



Solution. First, the domain of k is the interval $[-1, 3]$. We analyze this graph "anthropomorphically." We see that as x approaches -1 from the right (i.e., $x \to -1^+$) the graph tries to contain the point $(-1, 0)$ and it succeeds! So $\boxed{k \text{ is continuous from the right at } -1}$: $\lim_{x\to -1^+} k(x) = 0 = k(-1)$.

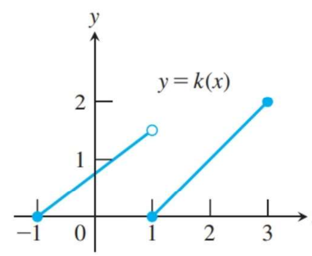
Exercise 2.5.4 (continued 1)



Solution (continued). As x approaches 3 from the left (i.e., $x \to 3^-$) the graph tries to contain the point $(3, 2)$ and it succeeds! So $\boxed{k \text{ is continuous from the left at } 3}$: $\lim_{x\to 3^-} k(x) = 2 = k(3)$.

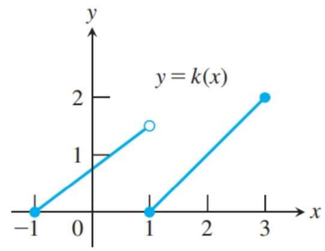
The graph of $y = k(x)$ on $(-1, 1)$ is a line and as x approaches any value c in this interval, the graph tries to pass through a point of the form $(c, f(c))$ and succeeds. So $\boxed{k \text{ is continuous at each of the points in } (-1, 1)}$: $\lim_{x\to c} k(x) = k(c)$ for $c \in (-1, 1)$.

Exercise 2.5.4 (continued 2)



Solution (continued). Similarly, the graph of $y = k(x)$ on $(1, 3)$ is a line and as x approaches any value c in this interval, the graph tries to pass through a point of the form $(c, f(c))$ and succeeds. So $\boxed{k \text{ is continuous at each of the interior points in } (1, 3)}$: $\lim_{x\to c} k(x) = k(c)$ for $c \in (1, 3)$.

Exercise 2.5.4 (continued 3)



Solution (continued). Now as x approaches 1 from the left (i.e., $x \to 1^-$) the graph tries to contain the point $(1, 3/2)$. As x approaches 1 from the right (i.e., $x \to 1^+$) the graph tries to contain the point $(1, 0)$ (and it succeeds). So the two-sided limit as x approaches 1 does not exist and hence $\boxed{k \text{ is not continuous at } x = 1}$.

So $\boxed{k \text{ is continuous on the set } [-1, 1) \cup (1, 3]}$. $\square$

Example 2.5.A

Example 2.5.A. Consider the piecewise defined function

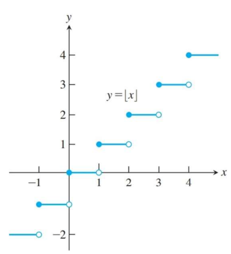
$$f(x) = \begin{cases} x & \text{if } x \in (-\infty, 0) \\ 0 & \text{if } x = 0 \\ x^2 & \text{if } x \in (0, \infty). \end{cases}$$

Is f continuous at $x = 0$?

Solution. Since $x = 0$ is an interior point of the domain of f, we apply part (a) of the Continuity Test. First, $f(0) = 0$ exists. To address $\lim_{x\to 0} f(x)$, we use the Relation Between One-Sided and Two-Sided Limits (Theorem 2.6). We have $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = (0) = 0$ by Theorem 2.2 (since x is a polynomial), and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = (0)^2 = 0$ by Theorem 2.2 (since x^2 is a polynomial). So, by Theorem 2.6, $\lim_{x\to 0} f(x) = 0$. Finally, $\lim_{x\to 0} f(x) = 0 = f(0)$, so by the Continuity Test, $\boxed{f \text{ is continuous at } x = 0}$. $\square$

Example 2.5.4

Example 2.5.4. Discuss the discontinuities of (a) $g(x) = \text{int}\, x = \lfloor x \rfloor$ (this is Example 2.5.4) and (b) $f(x) = \dfrac{|x|}{x}$.

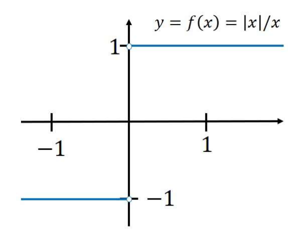


Solution. (a) Notice that at each integer n we have $\lim_{x\to n^-} \lfloor x \rfloor = n - 1$ and $\lim_{x\to n^+} \lfloor x \rfloor = n$. So $\boxed{\text{at each integer } n,\ \lfloor x \rfloor \text{ has a jump discontinuity}}$. Next, for n and integer, $\lfloor x \rfloor$ is constant on the interval $(n, n+1)$ and so the limit at such values exists (by Example 2.3.3(b), say) and equals the function value. So $\boxed{\lfloor x \rfloor \text{ is continuous at all non-integer values}}$. $\square$

Example 2.5.4 (continued)

Solution. (b) Notice that for $x > 0$ we have $f(x) = \dfrac{|x|}{x} = 1$, and for $x < 0$ we have $f(x) = \dfrac{|x|}{x} = -1$. So for $c > 0$ we have $\lim_{x\to c} f(x) = \lim_{x\to c} 1 = 1 = f(c)$, and for $c < 0$ we have $\lim_{x\to c} f(x) = \lim_{x\to c} -1 = -1 = f(c)$ (both by Example 2.3.3(b); notice that for $c \neq 0$, there is an interval containing c on which f is constant). So $\boxed{f(x) = |x|/x \text{ is continuous for } x \neq 0}$.

For $c = 0$, notice that
$\lim_{x\to 0^-} |x|/x = \lim_{x\to 0^-}(-1) = -1$ and
$\lim_{x\to 0^+} |x|/x = \lim_{x\to 0^+}(1) = 1$.
So, by definition,
$\boxed{f(x) = |x|/x \text{ has a jump discontinuity at } x = 0}$.
$\square$



Exercise 2.5.42

Exercise 2.5.42. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.

Solution. Notice that

$$\begin{aligned}\lim_{t\to 2} \frac{t^2+3t-10}{t-2} &= \lim_{t\to 2} \frac{(t-2)(t+5)}{t-2} \\ &= \lim_{t\to 2} t+5 \text{ by Dr. Bob's Limit Theorem,} \\ &\quad \text{Theorem 2.2.A} \\ &= (2)+5 = 7 \text{ by Theorem 2.2.}\end{aligned}$$

Since this limit exists, but h is not defined at $t = 2$ then h has a removable discontinuity at $t = 2$. If we $\boxed{\text{redefine } h(2) = 7}$, then we get the continuous extension of h, as desired. □

Exercise 2.5.72

Exercise 2.5.72. In Exercise 2.5.71, it is shown that f is continuous at c if and only if $\lim_{h\to 0} f(h+c) = f(c)$. Use this, Example 2.2.11(a)(b), in which it is shown that $\lim_{\theta\to 0} \sin\theta = 0$ and $\lim_{\theta\to 0} \cos\theta = 1$, and the identities

$$\sin(h+c) = \sin h \cos c + \cos h \sin c \text{ and } \cos(h+c) = \cos h \cos c - \sin h \cos c$$

to prove that both $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at every point $x = c$.

Solution. First, let c be an arbitrary point. We have

$$\begin{aligned}\lim_{h\to 0} \sin(c+h) &= \lim_{h\to 0}(\sin h \cos c + \cos h \sin c) \text{ by the addition formula} \\ &= \lim_{h\to 0}(\sin h \cos c) + \lim_{h\to 0}(\cos h \sin c) \text{ by the} \\ &\quad \text{Sum Rule, Theorem 2.1(1)} \\ &= \cos c \lim_{h\to 0}(\sin h) + \sin c \lim_{h\to 0}(\cos h) \text{ by the} \\ &\quad \text{Constant Multiple Rule, Theorem 2.1(3)}\end{aligned}$$

Exercise 2.5.72 (continued 1)

Solution (continued). ...

$$\begin{aligned}\lim_{h\to 0} \sin(c+h) &= \cos c \lim_{h\to 0}(\sin h) + \sin c \lim_{h\to 0}(\cos h) \\ &= (\cos c)(0) + (\sin c)(1) \text{ by Example 2.2.11(a) and (b)} \\ &= \sin c.\end{aligned}$$

So by Exercise 2.5.71, $f(x) = \sin x$ is continuous at every point $x = c$.

We also have

$$\begin{aligned}\lim_{h\to 0} \cos(c+h) &= \lim_{h\to 0}(\cos h \cos c - \sin h \cos c) \text{ by the addition formula} \\ &= \lim_{h\to 0}(\cos h \cos c) - \lim_{h\to 0}(\sin h \sin c) \text{ by the} \\ &\quad \text{Difference Rule, Theorem 2.1(2)} \\ &= \cos c \lim_{h\to 0}(\cos h) - \sin c \lim_{h\to 0}(\sin h) \text{ by the} \\ &\quad \text{Constant Multiple Rule, Theorem 2.1(3)}\end{aligned}$$

Exercise 2.5.72 (continued 2)

Exercise 2.5.72. In Exercise 2.5.71, it is shown that f is continuous at c if and only if $\lim_{h\to 0} f(h+c) = f(c)$. Use this, Example 2.2.11(a)(b), in which it is shown that $\lim_{\theta\to 0} \sin\theta = 0$ and $\lim_{\theta\to 0} \cos\theta = 1$, and the identities

$$\sin(h+c) = \sin h \cos c + \cos h \sin c \text{ and } \cos(h+c) = \cos h \cos c - \sin h \cos c$$

to prove that both $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at every point $x = c$.

Solution (continued). ...

$$\begin{aligned}\lim_{h\to 0} \cos(c+h) &= \cos c \lim_{h\to 0}(\cos h) - \sin c \lim_{h\to 0}(\sin h) \\ &= (\cos c)(1) - (\sin c)(0) \text{ by Example 2.2.11(a) and (b)} \\ &= \cos c.\end{aligned}$$

So by Exercise 2.5.71, $g(x) = \cos x$ is continuous at every point $x = c$. □

Exercise 2.5.26

Exercise 2.5.26. Consider the function $h(x) = \sqrt[4]{3x-1}$. At what points is f continuous and why? Explain by considering interior points and endpoints of the domain.

Solution. The domain of $h(x) = \sqrt[4]{3x-1}$ is all x satisfying $3x - 1 \geq 0$; that is, all $x \geq 1/3$. Define $g(x) = \sqrt[4]{x}$ and $f(x) = 3x - 1$, so that $h = g \circ f$: $h(x) = \sqrt[4]{3x-1} = \sqrt[4]{f(x)} = g(f(x))$. For c an interior point of the domain of h (so $c > 1/3$) we have that $f(x) = 3x - 1$ is continuous at c by Theorem 2.5.A, since f is a polynomial function. For such c, $f(c) = 3c - 1 > 0$. Now $g(x) = \sqrt[4]{x}$ is defined on an open interval containing $f(c)$ (say on the interval $(f(c)/2, f(c)+1)$ since this interval only contains positive numbers), so by Theorem 2.8(7), "Roots," g is continuous at $f(c)$. So, by Compositions of Continuous Functions (Theorem 2.9) $h = g \circ f$, or

$\boxed{h(x) = g(f(x)) = \sqrt[4]{3x-1}, \text{ is continuous at all interior points } c > 1/3}$

of the domain of h.

Exercise 2.5.26 (continued)

Solution (continued). For the left-hand endpoint $c = 1/3$ of the domain of h, we use the Continuity Test. Now $3x - 1 \geq 0$ on an open interval of the form $(1/3, 1/3 + \delta)$ (we could take $\delta = 1$, for example), so by the Root Rule (Theorem 2.1(7)) applied to the one-sided limit $\lim_{x\to 1/3^+} h(x)$ we have

$$\lim_{x\to 1/3^+} h(x) = \lim_{x\to 1/3^+} \sqrt[4]{3x-1} = \sqrt[4]{\lim_{x\to 1/3^+}(3x-1)} = \sqrt[4]{0} = 0 = h(1/3).$$

Notice that we must use the version of the Root Rule stated in these notes and not the version stated in the text book.

So, by the Continuity Test we have that

$\boxed{h(x) = g(f(x)) = \sqrt[4]{3x-1} \text{ is continuous at the left-hand endpoint } c = 1/3}$

of the domain of h.

We can say that $\boxed{h \text{ is continuous on its domain } [1/3, \infty)}$, with the understanding that we have continuity from the right at the endpoint $1/3$ of the domain. $\square$

Theorem 2.10

Theorem 2.10. Limits of Continuous Functions.
If g is continuous at the point b and $\lim_{x\to c} f(x) = b$, the

$$\lim_{x\to c} g(f(x)) = g(b) = g\left(\lim_{x\to c} f(x)\right).$$

Proof. Let $\varepsilon > 0$. Since g is continuous at b by hypothesis, then $\lim_{y\to b} g(y) = g(b)$. So by the (formal) definition of limit, there exists $\delta_1 > 0$ such that

$$0 < |y - b| < \delta_1 \text{ implies } |g(y) - g(b)| < \varepsilon.$$

Since $\lim_{x\to c} f(x) = b$ by hypothesis, then there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \text{ implies } |f(x) - b| < \delta_1$$

(here, δ_1 plays the role of an arbitrary positive $\varepsilon > 0$). Let $y = f(x)$.

Theorem 2.10 (continued)

Proof (continued). Then we have that

$$0 < |x - c| < \delta \text{ implies } |f(x) - b| < \delta_1 \text{ or } |y - b| < \delta_1 \text{ which implies}$$

$$|g(y) - g(b)| < \varepsilon \text{ or } |g(f(x)) - g(b)| < \varepsilon.$$

That is, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \text{ implies } |g(f(x)) - g(b)| < \varepsilon.$$

Therefore, by the definition of limit, we have $\lim_{x\to c} g(f(x)) = g(b) = g(\lim_{x\to c} f(x))$, as claimed. $\square$

Exercise 2.5.34

Exercise 2.5.34. Is the function continuous at the point being approached: $\lim_{t\to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right)$? Explain.

Solution. First, since $\tan t$ is continuous on its domain by Theorem 2.5.B then by the definition of continuity we have $\lim_{t\to 0} \tan t = \tan 0 = 0$; that is, $0 = \tan 0 = \tan(\lim_{t\to 0} t) = \lim_{t\to 0} \tan t$.

Second, since $\frac{\pi}{2}\cos u$ is continuous at $u = \tan 0 = 0$ (by Theorem 2.5.B and Theorem 2.8(4)), then by Limits of Continuous Functions (Theorem 2.10), we have $\lim_{t\to 0} \frac{\pi}{2}\cos(\tan t) = \frac{\pi}{2}\cos(\lim_{t\to 0} \tan t) = \frac{\pi}{2}\cos(0) = \frac{\pi}{2}$.

Third, since $\sin v$ is continuous at $v = \frac{\pi}{2}\cos(\tan 0) = \pi/2$ (by Theorem 2.5.B), then by Limits of Continuous Functions (Theorem 2.10), we have $\lim_{t\to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right) = \sin\left(\lim_{t\to 0} \frac{\pi}{2}\cos(\tan t)\right) = \sin\left(\frac{\pi}{2}\right) = 1$.

So $\lim_{t\to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right) = 1 = \sin\left(\frac{\pi}{2}\cos(\tan 0)\right)$ and so $\boxed{\text{yes}}$ the function is continuous at $t = 0$. □

Exercise 2.5.56

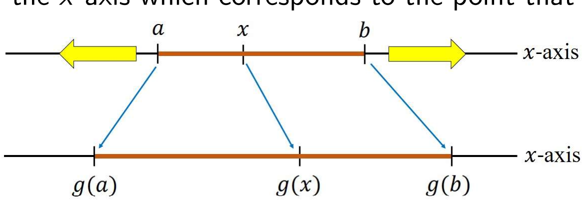
Exercise 2.5.56. Prove that the equation $\cos x = x$ has at least one solution. Give reasons for your answer.

Proof. Let $f(x) = \cos x - x$. Since $\cos x$ is continuous by Theorem 2.5.B and x is continuous by Theorem 2.5.A, then f is continuous by Theorem 2.8(2), "Differences." If we can find a and b such that $f(a) > 0$ and $f(b) < 0$, then the Intermediate Value Theorem will imply that there exists c between a and b such that $f(c) = 0$. We try $a = 0$ and $b = \pi/2$ (think about the graphs of $y = \cos x$ and $y = x$). Then
$f(a) = f(0) = \cos(0) - (0) = 1 - 0 = 1 > 0$ and
$f(b) = f(\pi/2) = \cos(\pi/2) - (\pi/2) = 0 - \pi/2 = -\pi/2 < 0$. Since y_0 is between $f(a) = 1$ and $f(b) = -\pi/2$, then by the Intermediate Value Theorem there exists $c \in [a, b] = [0, \pi/2]$ such that $f(c) = y_0 = 0$. That is, there is some $c \in [0, \pi/2]$ such that $f(c) = \cos(c) - (c) = 0$, or $\cos c = c$ as desired. □

Exercise 2.5.68

Exercise 2.5.68. Stretching a Rubber Band.
Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? Give mathematical reasons for your answer.

Solution. Let the rubber band lie on the interval $[a, b]$ on the x-axis of a Cartesian coordinate system. Label the points on the rubber band according to the x coordinate of the point on the x-axis where it lies (so the left end of the rubber band is labeled a and the right endpoint is labeled b). When the rubber band is stretched, let $g(x)$ represent the new coordinate on the x-axis which corresponds to the point that was originally at point x.



Exercise 2.5.68 (continued)

Exercise 2.5.68. Stretching a Rubber Band.
Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? Give mathematical reasons for your answer.

Solution (continued). Implicit in the physics here is that g is a continuous function (the rubber doesn't break, for example). Since the left end was moved to the left, then $g(a) < a$. Since the right end was moved to the right, then $g(b) > b$. Consider the function $f(x) = g(x) - x$ (this is the "signed distance" that the point moves to the right). Then f is continuous by Theorem 2.8(2), "Differences." Notice that $f(a) = g(a) - a < 0$ and $f(b) = g(b) - b > 0$. Since 0 is between $f(a) < 0$ and $f(b) > 0$ then by the Intermediate Value Theorem, there is $c \in [a, b]$ such that $f(c) = g(c) - c = 0$. That is, there is a point $x = c$ on the rubber band that is in its original position after the rubber band is stretched (i.e., $c = g(c)$). $\boxed{\text{Yes}}$, it is true. □