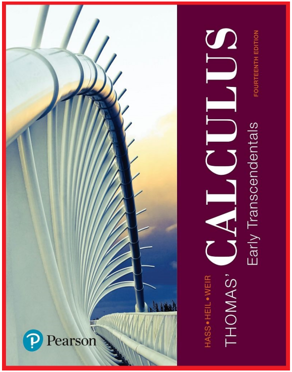
Calculus 1

Chapter 2. Limits and Continuity

2.5. Continuity—Examples and Proofs



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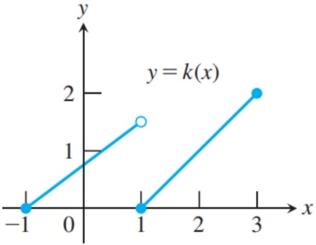

Solution. First, the domain of k is the interval $[-1, 3]$. We analyze this graph "anthropomorphically." We see that as x approaches -1 from the right (i.e., $x \to -1^+$) the graph tries to contain the point $(-1, 0)$ and it succeeds! So $\boxed{k \text{ is continuous from the right at } -1}$: $\lim_{x \to -1^+} k(x) = 0 = k(-1)$.

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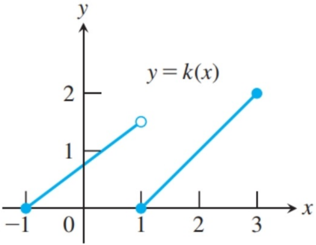

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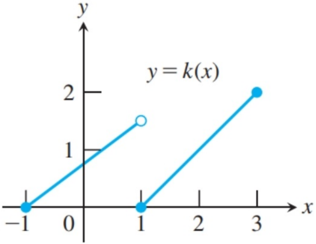

Solution (continued). As x approaches 3 from the left (i.e., $x \to 3^-$) the graph tries to contain the point $(3, 2)$ and it succeeds! So $\boxed{k \text{ is continuous from the left at } 3}$: $\lim_{x \to 3^-} k(x) = 2 = k(3)$.

The graph of $y = k(x)$ on $(-1, 1)$ is a line and as x approaches any value c in this interval, the graph tries to pass through a point of the form $(c, f(c))$ and succeeds. So $\boxed{k \text{ is continuous at each of the points in } (-1, 1)}$: $\lim_{x \to c} k(x) = k(c)$ for $c \in (-1, 1)$.

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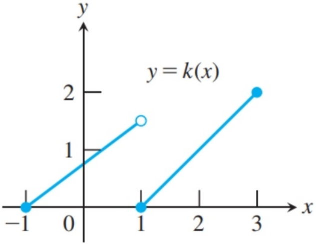

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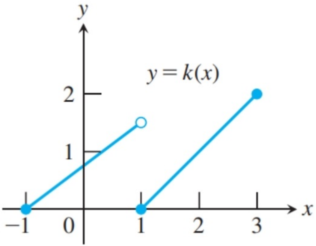
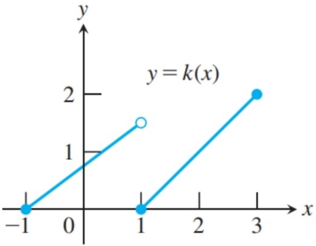


Solution (continued). Similarly, the graph of $y = k(x)$ on $(1, 3)$ is a line and as x approaches any value c in this interval, the graph tries to pass through a point of the form $(c, f(c))$ and succeeds. So

$\boxed{k \text{ is continuous at each of the interior points in } (1, 3)}$:

$\lim_{x \to c} k(x) = k(c)$ for $c \in (1, 3)$.

Exercise 2.5.4 (continued 3)



Solution (continued). Now as x approaches 1 from the left (i.e., $x \to 1^-$) the graph tries to contain the point $(1, 3/2)$. As x approaches 1 from the right (i.e., $x \to 1^+$) the graph tries to contain the point $(1, 0)$ (and it succeeds). So the two-sided limit as x approaches 1 does not exist and hence $\boxed{k \text{ is not continuous at } x = 1}$.

So $\boxed{k \text{ is continuous on the set } [-1, 1) \cup (1, 3]}$. □

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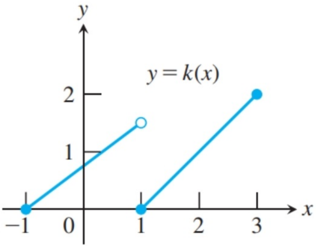

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Example 2.5.A

Example 2.5.A. Consider the piecewise defined function

$$f(x) = \begin{cases} x & \text{if } x \in (-\infty, 0) \\ 0 & \text{if } x = 0 \\ x^2 & \text{if } x \in (0, \infty). \end{cases}$$

Is f continuous at $x = 0$?

Solution. Since $x = 0$ is an interior point of the domain of f, we apply part (a) of the Continuity Test. First, $f(0) = 0$ exists.

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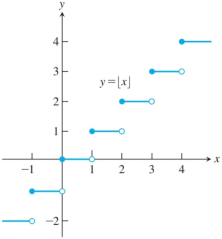
Example 2.5.4. Discuss the discontinuities of (a) $g(x) = \text{int}\, x = \lfloor x \rfloor$ (this is Example 2.5.4) and (b) $f(x) = \dfrac{|x|}{x}$.



Solution. **(a)** Notice that at each integer n we have $\lim_{x\to n^-} \lfloor x \rfloor = n - 1$ and $\lim_{x\to n^+} \lfloor x \rfloor = n$. So $\boxed{\text{at each integer } n,\ \lfloor x \rfloor \text{ has a jump discontinuity}}$.

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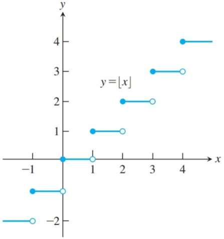

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Next, for n and integer, $\lfloor x \rfloor$ is constant on the interval $(n, n+1)$ and so the limit at such values exists (by Example 2.3.3(b), say) and equals the function value. So $\boxed{\lfloor x \rfloor \text{ is continuous at all non-integer values}}$. □

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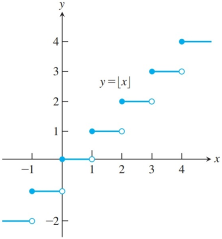

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Example 2.5.4 (continued)

Solution. **(b)** Notice that for $x > 0$ we have $f(x) = \dfrac{|x|}{x} = 1$, and for $x < 0$ we have $f(x) = \dfrac{|x|}{x} = -1$. So for $c > 0$ we have $\lim_{x\to c} f(x) = \lim_{x\to c} 1 = 1 = f(c)$, and for $c < 0$ we have $\lim_{x\to c} f(x) = \lim_{x\to c} -1 = -1 = f(c)$ (both by Example 2.3.3(b); notice that for $c \neq 0$, there is an interval containing c on which f is constant).
So $\boxed{f(x) = |x|/x \text{ is continuous for } x \neq 0}$.

For $c = 0$, notice that
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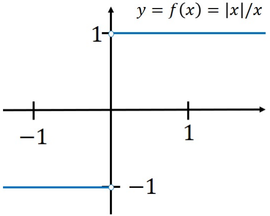

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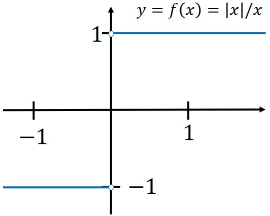

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Exercise 2.5.42. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.

Solution. Notice that

$$\begin{aligned} \lim_{t\to 2} \frac{t^2 + 3t - 10}{t - 2} &= \lim_{t\to 2} \frac{(t-2)(t+5)}{t-2} \\ &= \lim_{t\to 2} t + 5 \text{ by Dr. Bob's Limit Theorem,} \\ &\quad \text{Theorem 2.2.A} \\ &= (2) + 5 = 7 \text{ by Theorem 2.2.} \end{aligned}$$

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Exercise 2.5.72

Exercise 2.5.72. In Exercise 2.5.71, it is shown that f is continuous at c if and only if $\lim_{h\to 0} f(h+c) = f(c)$. Use this, Example 2.2.11(a)(b), in which it is shown that $\lim_{\theta\to 0}\sin\theta = 0$ and $\lim_{\theta\to 0}\cos\theta = 1$, and the identities

$$\sin(h+c) = \sin h\cos c + \cos h\sin c \text{ and } \cos(h+c) = \cos h\cos c - \sin h\cos c$$

to prove that both $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at every point $x = c$.

Solution. First, let c be an arbitrary point. We have

$$\begin{aligned}
\lim_{h\to 0}\sin(c+h) &= \lim_{h\to 0}(\sin h\cos c + \cos h\sin c) \text{ by the addition formula} \\
&= \lim_{h\to 0}(\sin h\cos c) + \lim_{h\to 0}(\cos h\sin c) \text{ by the} \\
&\quad \text{Sum Rule, Theorem 2.1(1)} \\
&= \cos c\lim_{h\to 0}(\sin h) + \sin c\lim_{h\to 0}(\cos h) \text{ by the} \\
&\quad \text{Constant Multiple Rule, Theorem 2.1(3)}
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\end{aligned}$$

Exercise 2.5.72 (continued 1)

Solution (continued). ...

$$\begin{aligned}\lim_{h\to 0}\sin(c+h) &= \cos c \lim_{h\to 0}(\sin h) + \sin c \lim_{h\to 0}(\cos h) \\ &= (\cos c)(0) + (\sin c)(1) \text{ by Example 2.2.11(a) and (b)} \\ &= \sin c.\end{aligned}$$

So by Exercise 2.5.71, $f(x) = \sin x$ is continuous at every point $x = c$.

We also have

$$\begin{aligned}\lim_{h\to 0}\cos(c+h) &= \lim_{h\to 0}(\cos h \cos c - \sin h \cos c) \text{ by the addition formula} \\ &= \lim_{h\to 0}(\cos h \cos c) - \lim_{h\to 0}(\sin h \sin c) \text{ by the} \\ &\quad \text{Difference Rule, Theorem 2.1(2)} \\ &= \cos c \lim_{h\to 0}(\cos h) - \sin c \lim_{h\to 0}(\sin h) \text{ by the} \\ &\quad \text{Constant Multiple Rule, Theorem 2.1(3)}\end{aligned}$$

Exercise 2.5.72 (continued 1)

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$$\begin{aligned}\lim_{h\to 0}\sin(c+h) &= \cos c\lim_{h\to 0}(\sin h)+\sin c\lim_{h\to 0}(\cos h)\\ &= (\cos c)(0)+(\sin c)(1) \text{ by Example 2.2.11(a) and (b)}\\ &= \sin c.\end{aligned}$$

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Exercise 2.5.72 (continued 2)

Exercise 2.5.72. In Exercise 2.5.71, it is shown that f is continuous at c if and only if $\lim_{h\to 0} f(h+c) = f(c)$. Use this, Example 2.2.11(a)(b), in which it is shown that $\lim_{\theta\to 0}\sin\theta = 0$ and $\lim_{\theta\to 0}\cos\theta = 1$, and the identities

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$$\begin{aligned}\lim_{h\to 0}\cos(c+h) &= \cos c\lim_{h\to 0}(\cos h) - \sin c\lim_{h\to 0}(\sin h)\\ &= (\cos c)(1) - (\sin c)(0) \text{ by Example 2.2.11(a) and (b)}\\ &= \cos c.\end{aligned}$$

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Exercise 2.5.26

Exercise 2.5.26. Consider the function $h(x) = \sqrt[4]{3x-1}$. At what points is f continuous and why? Explain by considering interior points and endpoints of the domain.

Solution. The domain of $h(x) = \sqrt[4]{3x-1}$ is all x satisfying $3x - 1 \geq 0$; that is, all $x \geq 1/3$. Define $g(x) = \sqrt[4]{x}$ and $f(x) = 3x - 1$, so that $h = g \circ f$: $h(x) = \sqrt[4]{3x-1} = \sqrt[4]{f(x)} = g(f(x))$.

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Solution. The domain of $h(x) = \sqrt[4]{3x-1}$ is all x satisfying $3x - 1 \geq 0$; that is, all $x \geq 1/3$. Define $g(x) = \sqrt[4]{x}$ and $f(x) = 3x - 1$, so that $h = g \circ f$: $h(x) = \sqrt[4]{3x-1} = \sqrt[4]{f(x)} = g(f(x))$. For c an interior point of the domain of h (so $c > 1/3$) we have that $f(x) = 3x - 1$ is continuous at c by Theorem 2.5.A, since f is a polynomial function. For such c, $f(c) = 3c - 1 > 0$. Now $g(x) = \sqrt[4]{x}$ is defined on an open interval containing $f(c)$ (say on the interval $(f(c)/2, f(c) + 1)$ since this interval only contains positive numbers), so by Theorem 2.8(7), "Roots," g is continuous at $f(c)$. So, by Compositions of Continuous Functions (Theorem 2.9) $h = g \circ f$, or

$$\boxed{h(x) = g(f(x)) = \sqrt[4]{3x-1}, \text{ is continuous at all interior points } c > 1/3}$$

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Exercise 2.5.26 (continued)

Solution (continued). For the left-hand endpoint $c = 1/3$ of the domain of h, we use the Continuity Test. Now $3x - 1 \geq 0$ on an open interval of the form $(1/3, 1/3 + \delta)$ (we could take $\delta = 1$, for example), so by the Root Rule (Theorem 2.1(7)) applied to the one-sided limit $\lim_{x \to 1/3^+} h(x)$ we have

$$\lim_{x \to 1/3^+} h(x) = \lim_{x \to 1/3^+} \sqrt[4]{3x - 1} = \sqrt[4]{\lim_{x \to 1/3^+} (3x - 1)} = \sqrt[4]{0} = 0 = h(1/3).$$

Notice that we must use the version of the Root Rule stated in these notes and not the version stated in the text book.

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We can say that h is continuous on its domain $[1/3, \infty)$, with the understanding that we have continuity from the right at the endpoint $1/3$ of the domain. □

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Theorem 2.10

Theorem 2.10. Limits of Continuous Functions.
If g is continuous at the point b and $\lim_{x \to c} f(x) = b$, the

$$\lim_{x \to c} g(f(x)) = g(b) = g\left(\lim_{x \to c} f(x)\right).$$

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That is, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

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Exercise 2.5.34

Exercise 2.5.34. Is the function continuous at the point being approached: $\lim_{t \to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right)$? Explain.

Solution. First, since $\tan t$ is continuous on its domain by Theorem 2.5.B then by the definition of continuity we have $\lim_{t\to 0} \tan t = \tan 0 = 0$; that is, $0 = \tan 0 = \tan(\lim_{t\to 0} t) = \lim_{t\to 0} \tan t$.

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Second, since $\frac{\pi}{2}\cos u$ is continuous at $u = \tan 0 = 0$ (by Theorem 2.5.B and Theorem 2.8(4)), then by Limits of Continuous Functions (Theorem 2.10), we have $\lim_{t\to 0} \frac{\pi}{2}\cos(\tan t) = \frac{\pi}{2}\cos\left(\lim_{t\to 0} \tan t\right) = \frac{\pi}{2}\cos(0) = \frac{\pi}{2}$.

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Third, since $\sin v$ is continuous at $v = \frac{\pi}{2}\cos(\tan 0) = \pi/2$ (by Theorem 2.5.B), then by Limits of Continuous Functions (Theorem 2.10), we have $\lim_{t\to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right) = \sin\left(\lim_{t\to 0} \frac{\pi}{2}\cos(\tan t)\right) = \sin\left(\frac{\pi}{2}\right) = 1$.

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Exercise 2.5.56

Exercise 2.5.56. Prove that the equation $\cos x = x$ has at least one solution. Give reasons for your answer.

Proof. Let $f(x) = \cos x - x$. Since $\cos x$ is continuous by Theorem 2.5.B and x is continuous by Theorem 2.5.A, then f is continuous by Theorem 2.8(2), "Differences."

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$f(a) = f(0) = \cos(0) - (0) = 1 - 0 = 1 > 0$ and
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Exercise 2.5.68

Exercise 2.5.68. Stretching a Rubber Band.
Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? Give mathematical reasons for your answer.

Solution. Let the rubber band lie on the interval $[a, b]$ on the x-axis of a Cartesian coordinate system. Label the points on the rubber band according to the x coordinate of the point on the x-axis where it lies (so the left end of the rubber band is labeled a and the right endpoint is labeled b).

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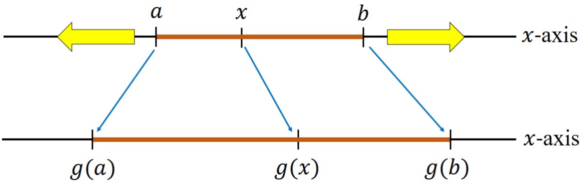

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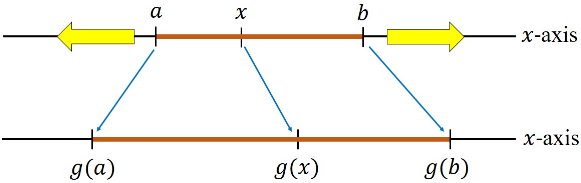

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Solution (continued). Implicit in the physics here is that g is a continuous function (the rubber doesn't break, for example). Since the left end was moved to the left, then $g(a) < a$. Since the right end was moved to the right, then $g(b) > b$. Consider the function $f(x) = g(x) - x$ (this is the "signed distance" that the point moves to the right). Then f is continuous by Theorem 2.8(2), "Differences." Notice that $f(a) = g(a) - a < 0$ and $f(b) = g(b) - b > 0$. Since 0 is between $f(a) < 0$ and $f(b) > 0$ then by the Intermediate Value Theorem, there is $c \in [a, b]$ such that $f(c) = g(c) - c = 0$. That is, there is a point $x = c$ on the rubber band that is in its original position after the rubber band is stretched (i.e., $c = g(c)$). **Yes**, it is true. □

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