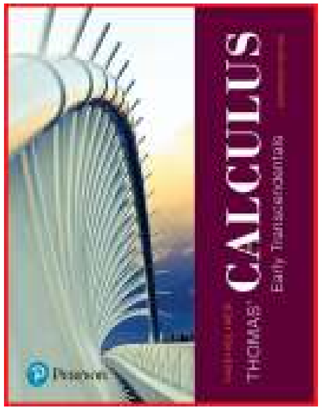
Calculus 1

Chapter 2. Limits and Continuity

2.6. Limits Involving Infinity; Asymptotes of Graphs—Examples and Proofs



Example 2.6.1(a)

Example 2.6.1(a). Prove that $\lim_{x\to\infty} \frac{1}{x} = 0$.

Proof. First, notice that with $P = 0$ we have that the domain of f contains the interval $(P, \infty) = (0, \infty)$. Let $\varepsilon > 0$ be given. [We must find a number M such that for all $x > M$ implies $\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \varepsilon$. The implication will hold if $M = 1/\varepsilon$ or any larger positive number (see Figure 2.50).]
Suppose $x > M = 1/\varepsilon$ (notice then that x is positive). This implies $0 < 1/x < 1/M = \varepsilon$, or $0 < \frac{1}{x} = \left|\frac{1}{x} - 0\right| = |f(x) - L| < \varepsilon$.
Therefore $\lim_{x\to\infty} \frac{1}{x} = 0$, as claimed. □

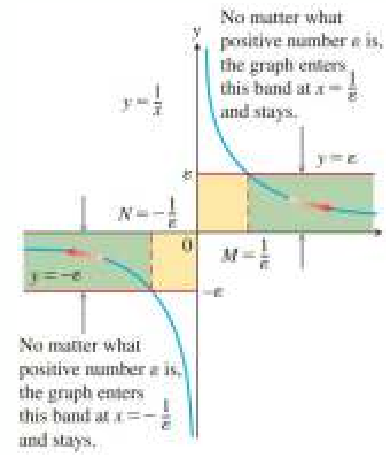


Figure 2.50

Exercise 2.6.14

Exercise 2.6.14. For the rational function $f(x) = \frac{2x^3+7}{x^3-x^2+x+7}$, find the limit as (a) $x \to \infty$, and (b) $x \to -\infty$. Justify your computations with Theorem 2.12.

Solution. We can evaluate both limits by the same process. We have

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{2x^3+7}{x^3-x^2+x+7} \text{ by the definition of } f$$

$$= \lim_{x\to\pm\infty} \frac{2x^3+7}{x^3-x^2+x+7}\left(\frac{1/x^3}{1/x^3}\right) \text{ dividing the numerator and denominator by the highest power of } x \text{ in the denominator}$$

$$= \lim_{x\to\pm\infty} \frac{(2x^3+7)/x^3}{(x^3-x^2+x+7)/x^3}$$

$$= \lim_{x\to\pm\infty} \frac{2x^3/x^3+7/x^3}{x^3/x^3-x^2/x^3+x/x^3+7/x^3}$$

Exercise 2.6.14 (continued 1)

Solution (continued).

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{2x^3/x^3+7/x^3}{x^3/x^3-x^2/x^3+x/x^3+7/x^3}$$

$$= \lim_{x\to\pm\infty} \frac{2+7/x^3}{1-1/x+1/x^2+7/x^3} \text{ since } x \to \pm\infty \text{ then we can assume that } x \neq 0$$

$$= \frac{\lim_{x\to\pm\infty}(2+7/x^3)}{\lim_{x\to\pm\infty}(1-1/x+1/x^2+7/x^3)} \text{ by the Quotient Rule (Theorem 2.12(5)), assuming the denominator is not 0}$$

$$= \frac{\lim_{x\to\pm\infty} 2 + \lim_{x\to\pm\infty} 7/x^3}{\lim_{x\to\pm\infty} 1 - \lim_{x\to\pm\infty} 1/x + \lim_{x\to\pm\infty} 1/x^2 + \lim_{x\to\pm\infty} 7/x^3)} \text{ by the Sum and Difference Rules (Theorem 2.12(1 and 2))}$$

$$= \frac{\lim_{x\to\pm\infty} 2 + 7\lim_{x\to\pm\infty} 1/x^3}{\lim_{x\to\pm\infty} 1 - \lim_{x\to\pm\infty} 1/x + \lim_{x\to\pm\infty} 1/x^2 + 7\lim_{x\to\pm\infty} 1/x^3)}$$

Exercise 2.6.14 (continued 2)

Solution (continued).

$$= \frac{\lim_{x\to\pm\infty} 2 + 7\lim_{x\to\pm\infty} 1/x^3}{\lim_{x\to\pm\infty} 1 - \lim_{x\to\pm\infty} 1/x + \lim_{x\to\pm\infty} 1/x^2 + 7\lim_{x\to\pm\infty} 1/x^3}$$

by the Constant Multiple Rule (Theorem 2.12(4))

$$= \frac{\lim_{x\to\pm\infty} 2 + 7\left(\lim_{x\to\pm\infty} 1/x\right)^3}{\lim_{x\to\pm\infty} 1 - \lim_{x\to\pm\infty} 1/x + \left(\lim_{x\to\pm\infty} 1/x\right)^2 + 7\left(\lim_{x\to\infty} 1/x\right)^3}$$

by the Power Rule (Theorem 2.12(6))

$$= \frac{(2) + 7(0)^3}{(1) - (0) + (0)^2 + 7(0)^3} \text{ by Example 2.6.1}$$

$$= \frac{2}{1} = \boxed{2}. \qquad \square$$

Exercise 2.6.36

Exercise 2.6.36. Evaluate $\lim\limits_{x\to-\infty} \dfrac{4-3x^3}{\sqrt{x^6+9}}$ by dividing the numerator and denominator by the (effective) highest power of x in the denominator. Justify your computations with Theorem 2.12.

Solution. We have a square root of x^6 in the denominator, so the "effective" highest power of x in the denominator is 3 (think what happens when x is really large: x^6+9 is about the same size as x^6 and $\sqrt{x^6+9}$ is about the same size as x^3). We have

$$\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}} = \lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}}\left(\frac{1/x^3}{1/x^3}\right)$$

dividing the numerator and denominator by the effective highest power of x in the denominator

$$= \lim_{x\to-\infty} \frac{(4-3x^3)/x^3}{(\sqrt{x^6+9})/x^3}$$

Exercise 2.6.36 (continued 1)

Solution (continued).

$$= \lim_{x\to-\infty} \frac{(4-3x^3)/x^3}{(\sqrt{x^6+9})/x^3} = \lim_{x\to-\infty} \frac{(4-3x^3)/x^3}{(\sqrt{x^6+9})/(-\sqrt{x^6})}$$

since $\sqrt{x^6} = |x^3| = -x^3$ for x negative

$$= \lim_{x\to-\infty} \frac{(4-3x^3)/x^3}{-\sqrt{(x^6+9)/x^6}} = \lim_{x\to-\infty} \frac{4/x^3 - 3x^3/x^3}{-\sqrt{x^6/x^6 + 9/x^6}}$$

$$= \lim_{x\to-\infty} \frac{4/x^3 - 3}{-\sqrt{1+9/x^6}}$$

since $x\to-\infty$ then we can assume that $x \neq 0$

$$= \frac{\lim_{x\to-\infty}(4/x^3-3)}{\lim_{x\to-\infty}(-\sqrt{1+9/x^6})}$$

by the Quotient Rule (Theorem 2.12(5)), assuming the denominator is not 0

Exercise 2.6.36 (continued 2)

Solution (continued).

$$= \frac{\lim_{x\to-\infty}(4/x^3) - \lim_{x\to-\infty} 3}{-\lim_{x\to-\infty}(\sqrt{1+9/x^6})}$$

by the Difference Rule and the Constant Multiple Rule, Theorem 2.12(2 and 4)

$$= \frac{\lim_{x\to-\infty}(4/x^3) - \lim_{x\to-\infty} 3}{-\sqrt{\lim_{x\to-\infty}(1+9/x^6)}}$$

by the Root Rule, Theorem 2.12(7)

$$= \frac{4\lim_{x\to-\infty}(1/x^3) - \lim_{x\to-\infty} 3}{-\sqrt{\lim_{x\to-\infty}(1) + 9\lim_{x\to-\infty}(1/x^6)}}$$

by the Sum Rule and Constant Multiple Rule, Theorem 2.12(1 and 4)

$$= \frac{4\left(\lim_{x\to-\infty} 1/x\right)^3 - \lim_{x\to-\infty} 3}{-\sqrt{\lim_{x\to-\infty}(1) + 9\left(\lim_{x\to-\infty} 1/x\right)^6}}$$

by the Power Rule, Theorem 2.12(6)

Exercise 2.6.36 (continued 3)

Exercise 2.6.36. Evaluate $\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}}$ by dividing the numerator and denominator by the (effective) highest power of x in the denominator. Justify your computations with Theorem 2.12.

Solution (continued).

$$\begin{aligned}\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}} &= \frac{4\,(\lim_{x\to-\infty} 1/x)^3 - \lim_{x\to-\infty} 3}{-\sqrt{\lim_{x\to-\infty}(1) + 9\,(\lim_{x\to-\infty} 1/x)^6}}\\ &= \frac{4(0)^3-(3)}{-\sqrt{(1)+9(0)^6}} \text{ by Example 2.6.1}\\ &= \frac{-3}{-1} = \boxed{3}. \qquad \square\end{aligned}$$

Exercise 2.6.68

Exercise 2.6.68. Find the horizontal asymptote(s) of the graph of $y = \frac{2x}{x+1}$. Justify your computations with Theorem 2.12.

Solution. By definition of horizontal asymptote, we are led to consider consider $\lim_{x\to\pm\infty} \frac{2x}{x+1}$. We have

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{2x}{x+1} &= \lim_{x\to\pm\infty} \frac{2x}{x+1}\left(\frac{1/x}{1/x}\right) \text{ dividing the numerator and denominator by the highest power of } x \text{ in the denominator}\\ &= \lim_{x\to\pm\infty} \frac{(2x)(1/x)}{(x+1)(1/x)} = \lim_{x\to\pm\infty} \frac{(2x/x)}{(x/x+1/x)}\\ &= \lim_{x\to\pm\infty} \frac{2}{1+1/x} \text{ since } x\to\pm\infty \text{ then we can assume that } x\neq 0\end{aligned}$$

Exercise 2.6.68 (continued)

Solution (continued).

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{2x}{x+1} &= \lim_{x\to\pm\infty} \frac{2}{1+1/x}\\ &= \frac{\lim_{x\to\pm\infty} 2}{\lim_{x\to\pm\infty}(1+1/x)} \text{ by the Quotient Rule (Theorem 2.12(5)), assuming the denominator is not 0}\\ &= \frac{\lim_{x\to\pm\infty} 2}{\lim_{x\to\pm\infty}(1) + \lim_{x\to\pm\infty}(1/x)} \text{ by the Sum Rule, Theorem 2.12(1)}\\ &= \frac{(2)}{(1)+(0)} = 2 \text{ by Example 2.6.1.}\end{aligned}$$

Since $\lim_{x\to\pm\infty} \frac{2x}{x+1} = 2$, then $\boxed{y = 2 \text{ is a horizontal asymptote}}$ of the graph of $y = \frac{2x}{x+1}$. $\square$

Example 2.6.4

Example 2.6.4. Find the horizontal asymptote(s) of the graph of $y = \frac{x^3-2}{|x|^3+1}$. Justify your computations with Theorem 2.12.

Solution. A rational function can have only one horizontal asymptote. Since we are not given a rational function (because of the presence of the absolute value), then we consider $x\to\infty$ and $x\to-\infty$ separately. We divide the numerator and denominator by the highest (effective) power of x in the denominator. We have

$$\begin{aligned}\lim_{x\to\infty} \frac{x^3-2}{|x|^3+1} &= \lim_{x\to\infty} \frac{x^3-2}{|x|^3+1}\left(\frac{1/x^3}{1/x^3}\right)\\ &= \lim_{x\to\infty} \frac{(x^3-2)(1/x^3)}{(|x|^3+1)(1/x^3)} = \lim_{x\to\infty} \frac{x^3/x^3 - 2/x^3}{x^3/x^3+1/x^3} \text{ since } x\to\infty \text{ then we can assume that } x \text{ is positive so that } |x|^3 = x^3\end{aligned}$$

Example 2.6.4 (continued 1)

Solution (continued).

$$\begin{aligned}\lim_{x\to\infty}\frac{x^3-2}{|x|^3+1} &= \lim_{x\to\infty}\frac{x^3/x^3-2/x^3}{x^3/x^3+1/x^3}\\ &= \lim_{x\to\infty}\frac{1-2/x^3}{1+1/x^3} \text{ since } x\to\infty \text{ then we can assume that } x\neq 0\\ &= \frac{\lim_{x\to\infty}(1-2/x^3)}{\lim_{x\to\infty}(1+1/x^3)} \text{ by the Quotient Rule (Theorem 2.12(5)), assuming the denominator is not 0}\\ &= \frac{\lim_{x\to\infty}(1)-\lim_{x\to\infty}(2/x^3)}{\lim_{x\to\infty}(1)+\lim_{x\to\infty}(1/x^3)} \text{ by the Sum Rule and the Difference Rule, Theorem 2.12(1 and 2)}\end{aligned}$$

Example 2.6.4 (continued 2)

Solution (continued).

$$\begin{aligned}\lim_{x\to\infty}\frac{x^3-2}{|x|^3+1} &= \frac{\lim_{x\to\infty}(1)-2\left(\lim_{x\to\infty}(1/x)\right)^3}{\lim_{x\to\infty}(1)+\left(\lim_{x\to\infty}(1/x)\right)^3} \text{ by the Constant Mult. Rule and the Power Rule, Theorem 2.12(4 and 6)}\\ &= \frac{(1)-2(0)^3}{(1)+(0)^3} = 1 \text{ by Example 2.6.1(a).}\end{aligned}$$

So the graph of $y = \dfrac{x^3-2}{|x|^3+1}$ has a $\boxed{\text{horizontal asymptote of } y = 1 \text{ as } x\to\infty}$.

Example 2.6.4 (continued 3)

Solution (continued). The computation is similar for $x\to-\infty$, except that for x negative we have $|x|^3=-x^3$. We have

$$\begin{aligned}\lim_{x\to-\infty}\frac{x^3-2}{|x|^3+1} &= \lim_{x\to-\infty}\frac{x^3-2}{|x|^3+1}\left(\frac{1/x^3}{1/x^3}\right)\\ &= \lim_{x\to-\infty}\frac{(x^3-2)(1/x^3)}{(|x|^3+1)(1/x^3)} = \lim_{x\to-\infty}\frac{x^3/x^3-2/x^3}{-x^3/x^3+1/x^3} \text{ since } x\to-\infty \text{ then we can assume that } x \text{ is negative so that } |x|^3=-x^3\\ &= \lim_{x\to-\infty}\frac{1-2/x^3}{-1+1/x^3} \text{ since } x\to-\infty \text{ then we can assume that } x\neq 0\\ &= \frac{\lim_{x\to-\infty}(1-2/x^3)}{\lim_{x\to-\infty}(-1+1/x^3)} \text{ by the Quotient Rule (Theorem 2.12(5)), assuming the denominator is not 0}\end{aligned}$$

Example 2.6.4 (continued 4)

Solution (continued).

$$\begin{aligned}\lim_{x\to-\infty}\frac{x^3-2}{|x|^3+1} &= \frac{\lim_{x\to-\infty}(1)-\lim_{x\to-\infty}(2/x^3)}{\lim_{x\to-\infty}(-1)+\lim_{x\to-\infty}(1/x^3)} \text{ by the Sum Rule and the Difference Rule, Theorem 2.12(1 and 2)}\\ &= \frac{\lim_{x\to-\infty}(1)-2\left(\lim_{x\to-\infty}(1/x)\right)^3}{\lim_{x\to-\infty}(-1)+\left(\lim_{x\to-\infty}(1/x)\right)^3} \text{ by Const. Mult. Rule and the Power Rule, Theorem 2.12(4 and 6)}\\ &= \frac{(1)-2(0)^3}{(-1)+(0)^3} = -1 \text{ by Example 2.6.1(b).}\end{aligned}$$

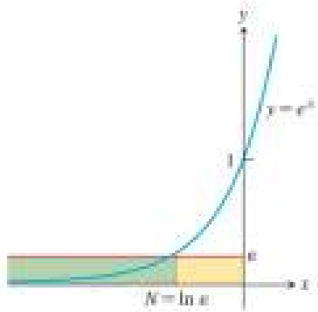
So the graph of $y = \dfrac{x^3-2}{|x|^3+1}$ has a $\boxed{\text{horizontal asymptote of } y = -1 \text{ as } x\to-\infty}$. □

Example 2.6.5

Example 2.6.5. Use the formal definition to prove $\lim_{x\to-\infty} e^x = 0$. Notice that this implies that $y = 0$ is a horizontal asymptote of $y = e^x$.

Proof. First, the domain of $f(x) = e^x$ is all of the real numbers $\mathbb{R}$, so it is defined on an interval of the form $(-\infty, P)$ (for any P). Next, let $\varepsilon > 0$. Choose $N = \ln\varepsilon$. If $x < N = \ln\varepsilon$ then $e^x < e^{\ln\varepsilon} = \varepsilon$ since e^x is an increasing function on the real numbers.
That is, if $x < N$ then
$|f(x) - 0| = |e^x - 0| = e^x < \varepsilon$.
Therefore, by definition,
$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} e^x = 0$,
as claimed. $\square$

Note. The choice of $N = \ln\varepsilon$ makes sense if we consider the graph of $y = e^x$:



Example 2.6.A

Example 2.6.A. Evaluate $\lim_{x\to\infty}\cos(1/x)$.

Solution. By Example 2.6.1(a) we have $\lim_{x\to\infty} 1/x = 0$, and by Exercise 2.5.72 we have that $\cos x$ is continuous at all points (in particular, it is continuous at 0). So by Theorem 2.6.A,

$$\lim_{x\to\infty}\cos(1/x) = \cos\left(\lim_{x\to\infty} 1/x\right) = \cos 0 = \boxed{1}.$$

$\square$

Example 2.6.8

Example 2.6.8. Use the Sandwich Theorem to find the horizontal asymptote of the curve $y = 2 + \dfrac{\sin x}{x}$.

Solution. First, $-1 \le \sin x \le 1$ for all real numbers. Let $g(x) = 2 - 1/x$, $f(x) = 2 + \dfrac{\sin x}{x}$, and $h(x) = 2 + 1/x$. Then $g(x) \le f(x) \le h(x)$ for all real numbers, except 0, and so these inequalities hold on $(-\infty, P) = (-\infty, 0)$ and $(P, \infty) = (0, \infty)$. Now
$\lim_{x\to\pm\infty} g(x) = \lim_{x\to\pm\infty}(2 - 1/x) = 2 - (0) = 2 = L$ and
$\lim_{x\to\pm\infty} h(x) = \lim_{x\to\pm\infty}(2 + 1/x) = 2 + (0) = 2 = L$, by Example 2.6.1.
So by Theorem 2.6.B, $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} 2 + \dfrac{\sin x}{x} = 2$. Therefore,
$\boxed{y = 2 \text{ is a horizontal asymptote}}$ of the graph of $y = 2 + \dfrac{\sin x}{x}$. $\square$

Exercise 2.6.92

Exercise 2.6.92. Evaluate (carefully!) $\lim_{x\to\infty}\left(\sqrt{x^2+x} - \sqrt{x^2-x}\right)$. Justify your computations.

Solution. We multiply by the conjugate of the given expression divided by itself (which is defined for x "sufficiently large," namely $x \ge 1$) in order to produce a quotient and try to use some of the techniques already introduced. We have

$$\begin{aligned} &\lim_{x\to\infty}\left(\sqrt{x^2+x} - \sqrt{x^2-x}\right) \\ = &\lim_{x\to\infty}\left(\sqrt{x^2+x} - \sqrt{x^2-x}\right)\left(\frac{\sqrt{x^2+x} + \sqrt{x^2-x}}{\sqrt{x^2+x} + \sqrt{x^2-x}}\right) \\ = &\lim_{x\to\infty}\frac{(\sqrt{x^2+x})^2 - (\sqrt{x^2-x})^2}{\sqrt{x^2+x} + \sqrt{x^2-x}} = \lim_{x\to\infty}\frac{(x^2+x) - (x^2-x)}{\sqrt{x^2+x} + \sqrt{x^2-x}} \\ = &\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+x} + \sqrt{x^2-x}} \end{aligned}$$

Exercise 2.6.92 (continued 1)

Solution (continued).

$$\begin{aligned}
&= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}} \\
&= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}\left(\frac{1/x}{1/x}\right) \text{ dividing the numerator and denominator by the effective highest power of } x \text{ in the denominator} \\
&= \lim_{x\to\infty} \frac{(2x)/x}{(\sqrt{x^2+x}+\sqrt{x^2-x})/x} = \lim_{x\to\infty} \frac{(2x)/x}{(\sqrt{x^2+x}+\sqrt{x^2-x})/\sqrt{x^2}} \\
&\text{since } x \to \infty \text{ then we can assume that } x \text{ is positive so that } \sqrt{x^2} = |x| = x \\
&= \lim_{x\to\infty} \frac{(2x)/x}{\sqrt{x^2+x}/\sqrt{x^2}+\sqrt{x^2-x}/\sqrt{x^2}}
\end{aligned}$$

Exercise 2.6.92 (continued 2)

Solution (continued).

$$\begin{aligned}
&= \lim_{x\to\infty} \frac{(2x)/x}{\sqrt{(x^2+x)/x^2}+\sqrt{(x^2-x)/x^2}} \\
&= \lim_{x\to\infty} \frac{(2x)/x}{\sqrt{x^2/x^2+x/x^2}+\sqrt{x^2/x^2-x/x^2}} \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{1+1/x}+\sqrt{1-1/x}} \text{ since } x\to\infty \text{ then we can assume that } x \neq 0 \\
&= \frac{\lim_{x\to\infty} 2}{\lim_{x\to\infty}(\sqrt{1+1/x}+\sqrt{1-1/x})} \text{ by the Quotient Rule (Theorem 2.12(5)), assuming the denominator is not 0} \\
&= \frac{\lim_{x\to\infty} 2}{\lim_{x\to\infty}\sqrt{1+1/x}+\lim_{x\to\infty}\sqrt{1-1/x})} \text{ by the Sum Rule, Theorem 2.12(1)}
\end{aligned}$$

Exercise 2.6.92 (continued 3)

Solution (continued).

$$\begin{aligned}
&= \frac{\lim_{x\to\infty} 2}{\lim_{x\to\infty}\sqrt{1+1/x}+\lim_{x\to\infty}\sqrt{1-1/x})} \\
&= \frac{\lim_{x\to\infty} 2}{\sqrt{\lim_{x\to\infty}(1+1/x)}+\sqrt{\lim_{x\to\infty}(1-1/x)})} \text{ by the Root Rule, Theorem 2.12(7) (notice that both } 1+1/x \text{ and } 1-1/x \text{ are nonnegative for } x \geq 1) \\
&= \frac{\lim_{x\to\infty} 2}{\sqrt{\lim_{x\to\infty}1+\lim_{x\to\infty}(1/x)}+\sqrt{\lim_{x\to\infty}1-\lim_{x\to\infty}(1/x)})} \text{ by the Sum and Difference Rules, Theorem 2.12(1 and 2)} \\
&= \frac{(2)}{\sqrt{(1)+(0)}+\sqrt{(1)-(0)})} = \frac{2}{1+1} = \boxed{1}. \qquad \square
\end{aligned}$$

Exercise 2.6.108

Exercise 2.6.108. Consider the rational function $y = \dfrac{x^2-1}{2x+4}$. Find the oblique asymptote.

Solution. First, we perform long division to get:

$$\begin{array}{r|l}
 & \;\; x/2 \; - \; 1 \\
\hline
2x+4 & x^2 \qquad\quad - \; 1 \\
 & x^2 + 2x \\
\hline
 & \quad - 2x - 1 \\
 & \quad - 2x - 4 \\
\hline
 & \qquad\qquad 3
\end{array}$$

So $y = \dfrac{x^2-1}{2x+4} = \dfrac{x}{2} - 1 + \dfrac{3}{2x+4}$ where $x/2-1$ is a linear term. If we show that $\lim_{x\to\pm\infty} 3/(2x+4) = 0$ then we can conclude that $y = x/2-1$ is the oblique asymptote for the graph of $y = (x^2-1)/(2x+4)$.

Exercise 2.6.108 (continued 1)

Solution (continued). Next,

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{3}{2x+4} &= \lim_{x\to\pm\infty} \frac{3}{2x+4}\left(\frac{1/x}{1/x}\right) \text{ dividing the numerator and} \\
&\quad \text{denominator by the highest power} \\
&\quad \text{of } x \text{ in the denominator} \\
&= \lim_{x\to\pm\infty} \frac{(3)(1/x)}{(2x+4)(1/x)} = \lim_{x\to\pm\infty} \frac{3/x}{2x/x+4/x} \\
&= \lim_{x\to\pm\infty} \frac{3/x}{2+4/x} \text{ since } x \to \pm\infty \text{ then we} \\
&\quad \text{can assume that } x \neq 0 \\
&= \frac{3\lim_{x\to\pm\infty} 1/x}{\lim_{x\to\pm\infty} 2 + 4\lim_{x\to\pm\infty} 1/x} \text{ by the Sum, Constant,} \\
&\quad \text{Multiple and Quotient Rules, Theorem 2.12(1, 4, \& 5)} \\
&= \frac{3(0)}{(2)+4(0)} = 0 \text{ by Example 2.6.1.}
\end{aligned}$$

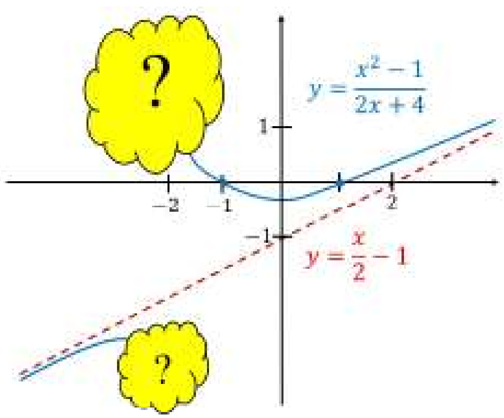
Exercise 2.6.108 (continued 2)

Exercise 2.6.108. Consider the rational function $y = \dfrac{x^2-1}{2x+4}$. Find the oblique asymptote.

Solution (continued). Since $y = \dfrac{x^2-1}{2x+4} = \dfrac{x}{2} - 1 + \dfrac{3}{2x+4}$ and $\displaystyle\lim_{x\to\pm\infty} \frac{3}{2x+4} = 0$, then $\boxed{y = \dfrac{x}{2} - 1 \text{ is an oblique asymptote}}$ of the graph of $y = \dfrac{x^2-1}{2x+4}$. Notice that the function $f(x) = \dfrac{x^2-1}{2x+4}$ is not defined at $x = -2$. With $y = \dfrac{x^2-1}{2x+4} = \dfrac{x}{2} - 1 + \dfrac{3}{2x+4}$, the term $\dfrac{3}{2x+4}$ is positive for x large and positive, and is negative for x large and negative. So the graph of $y = \dfrac{x^2-1}{2x+4}$ lies above the oblique asymptote $y = \dfrac{x}{2} - 1$ for x large and positive, and lies below the oblique asymptote for x large and negative.

Exercise 2.6.108 (continued 3)

Solution (continued). A crude graph of $y = \dfrac{x^2-1}{2x+4}$ which reflects the oblique asymptote (but does not reflect other subtle details of the graph) is as follows (we'll explore the graph in more detail later):

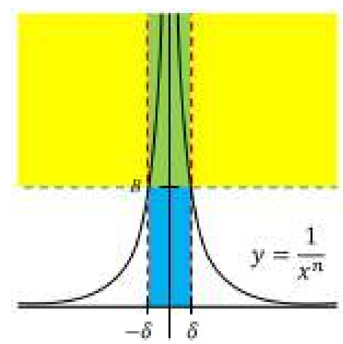


□

Example 2.6.B

Example 2.6.B. For n a positive even integer, prove that $\displaystyle\lim_{x\to 0} \frac{1}{x^n} = \infty$.

Solution. First, $f(x) = 1/x^n$ is defined for all x except 0, so there is an open interval containing $c = 0$ on which f is defined, except at $c = 0$ itself (say the interval $(-1,1)$). Let B be a positive real number. Choose $\delta = 1/B^{1/n}$. Then for $0 < |x - c| = |x - 0| = |x| < \delta = 1/B^{1/n}$, we have $1/|x| > B^{1/n}$ (since the function $1/x$ is decreasing for $x > 0$) and so $1/|x|^n > B$ (since the function x^n is increasing for $x \geq 0$). Since n is even, then $|x|^n = x^n$ and so we have $f(x) = 1/x^n = 1/|x|^n > B$. So, by definition, $\displaystyle\lim_{x\to 0} \frac{1}{x^n} = \infty$, as claimed.



□

Exercise 2.6.54

Exercise 2.6.54. Consider $f(x) = \dfrac{x}{x^2-1}$. Find (a) $\lim_{x\to 1^+} f(x)$, (b) $\lim_{x\to 1^-} f(x)$, (c) $\lim_{x\to -1^+} f(x)$, and (d) $\lim_{x\to -1^-} f(x)$.

Solution. First, $f(x) = \dfrac{x}{x^2-1}$ is a rational function of the form $f(x) = p(x)/q(x)$ where $p(x) = x$ and $q(x) = x^2-1$.
(a) We have $\lim_{x\to 1^+} p(x) = \lim_{x\to 1^+} x = 1 \neq 0$ and $\lim_{x\to 1^+} q(x) = \lim_{x\to 1^+}(x^2-1) = (1)^2 - 1 = 0$, by Theorem 2.2 for one-sided limits. So by Dr. Bob's Infinite Limits Theorem, $\displaystyle\lim_{x\to 1^+}\frac{p(x)}{q(x)} = \lim_{x\to 1^+}\frac{x}{x^2-1} = \pm\infty$; we just need to determine if the limit is $+\infty$ or $-\infty$. We do so by analyzing the sign of $\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)}$ for "appropriate" x (since $x \to 1^+$, then appropriate x are close to 1 and slightly greater than 1). For such x, we have x is positive (in fact, x is "close to" 1), $x-1$ is positive (since x is greater than 1; so $x-1$ is positive and "close to" 0), and $x+1$ is positive (in fact, x is "close to" 2).

Exercise 2.6.54 (continued 1)

Solution (continued). Combining the factors we can conclude the following little sign diagram (not an actual equation):
$\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)} \Rightarrow \dfrac{(+)}{(+)(+)} = +$. Since we know $\displaystyle\lim_{x\to 1^+}\frac{x}{x^2-1} = \pm\infty$ and we know for x close to 1 and slightly greater than 1 that $\dfrac{x}{x^2-1}$ is positive, then we conclude that $\boxed{\displaystyle\lim_{x\to 1^+}\frac{x}{x^2-1} = \infty}$. $\square$

(b) We have $\lim_{x\to 1^-} p(x) = \lim_{x\to 1^-} x = 1 \neq 0$ and $\lim_{x\to 1^-} q(x) = \lim_{x\to 1^-}(x^2-1) = (1)^2 - 1 = 0$, by Theorem 2.2 for one-sided limits. So by Dr. Bob's Infinite Limits Theorem, $\displaystyle\lim_{x\to 1^-}\frac{p(x)}{q(x)} = \lim_{x\to 1^-}\frac{x}{x^2-1} = \pm\infty$; we just need to determine if the limit is $+\infty$ or $-\infty$. We do so, again, by analyzing the sign of $\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)}$ for "appropriate" x (since $x \to 1^-$, then appropriate x are close to 1 and slightly less than 1).

Exercise 2.6.54 (continued 2)

Solution (continued). For such x, we have x is positive (in fact, x is "close to" 1), $x-1$ is negative (since x is less than 1; so $x-1$ is negative and "close to" 0), and $x+1$ is positive (in fact, x is "close to" 2). Combining the factors we can again conclude the following little sign diagram: $\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)} \Rightarrow \dfrac{(+)}{(-)(+)} = -$. Since we know $\displaystyle\lim_{x\to 1^-}\frac{x}{x^2-1} = \pm\infty$ and we know for x close to 1 and slightly less than 1 that $\dfrac{x}{x^2-1}$ is negative, then we conclude that $\boxed{\displaystyle\lim_{x\to 1^-}\frac{x}{x^2-1} = -\infty}$. $\square$

(c) We have $\lim_{x\to -1^+} p(x) = \lim_{x\to -1^+} x = -1 \neq 0$ and $\lim_{x\to -1^+} q(x) = \lim_{x\to -1^+}(x^2-1) = (-1)^2 - 1 = 0$, by Theorem 2.2 for one-sided limits. So by Dr. Bob's Infinite Limits Theorem, $\displaystyle\lim_{x\to -1^+}\frac{p(x)}{q(x)} = \lim_{x\to -1^+}\frac{x}{x^2-1} = \pm\infty$; we just need to determine if the limit is $+\infty$ or $-\infty$.

Exercise 2.6.54 (continued 3)

Solution (continued). We analyze the sign of $\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)}$ for "appropriate" x (since $x \to -1^+$, then appropriate x are close to -1 and slightly greater than -1). For such x, we have x is negative (in fact, x is "close to" -1), $x-1$ is negative (in fact, $x-1$ is "close to" -2), and $x+1$ is positive (since x is greater than -1; so $x+1$ is positive and "close to" 0). Combining the factors we get the sign diagram: $\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)} \Rightarrow \dfrac{(-)}{(-)(+)} = +$. Since we know $\displaystyle\lim_{x\to 1^-}\frac{x}{x^2-1} = \pm\infty$ and we know for x close to -1 and slightly greater than -1 that $\dfrac{x}{x^2-1}$ is positive, then we conclude that $\boxed{\displaystyle\lim_{x\to -1^+}\frac{x}{x^2-1} = \infty}$. $\square$

Exercise 2.6.54 (continued 4)

Solution (continued). (d) We have

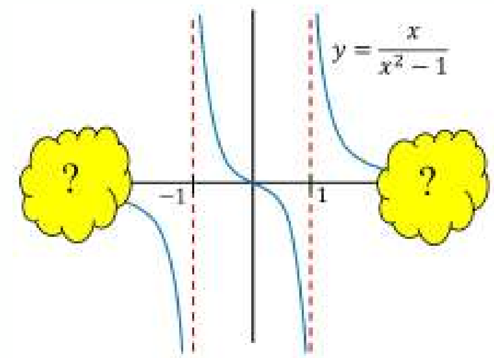
$\lim_{x\to -1^-} p(x) = \lim_{x\to -1^-} x = -1 \neq 0$ and $\lim_{x\to -1^-} q(x) = \lim_{x\to -1^-}(x^2-1) = (-1)^2 - 1 = 0$, by Theorem 2.2 for one-sided limits. So by Dr. Bob's Infinite Limits Theorem, $\displaystyle\lim_{x\to -1^-} \frac{p(x)}{q(x)} = \lim_{x\to -1^-} \frac{x}{x^2-1} = \pm\infty$; we just need to determine if the limit is $+\infty$ or $-\infty$. We analyze the sign of $\displaystyle\frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)}$ for "appropriate" x (since $x \to -1^-$, then appropriate x are close to -1 and slightly less than -1). For such x, we have x is negative (in fact, x is "close to" -1), $x-1$ is negative (in fact, $x-1$ is "close to" -2), and $x+1$ is negative (since x is less than -1; so $x+1$ is positive and "close to" 0). Combining the factors we get the sign diagram:

$$\frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)} \Rightarrow \frac{(-)}{(-)(-)} = -.$$

Exercise 2.6.54 (continued 5)

Solution (continued). Since we know $\displaystyle\lim_{x\to 1^-} \frac{x}{x^2-1} = \pm\infty$ and we know for x close to -1 and slightly less than -1 that $\displaystyle\frac{x}{x^2-1}$ is negative, then we conclude that $\displaystyle\boxed{\lim_{x\to -1^-} \frac{x}{x^2-1} = -\infty}$. □

Note. We know a lot about $f(x) = \dfrac{x}{x^2-1}$ and can get a reasonable graph of $y = f(x)$ by graphing its vertical asymptotes (notice also that $f(0) = 0$; we did not explore what happens for $|x|$ "large"):



Exercise 2.6.70

Exercise 2.6.70. Consider $y = f(x) = \dfrac{2x}{x^2-1}$. Find the domain, horizontal asymptote(s), vertical asymptotes, graph $y = f(x)$ in such a way as to reflect the asymptotic behavior, and find the range of f.

Solution. First, the domain of $y = f(x) = \dfrac{2x}{x^2-1} = \dfrac{2x}{(x+1)(x-1)}$ is all real x except for -1 and 1; the $\boxed{\text{domain is } (-\infty,-1)\cup(-1,1)\cup(1,\infty)}$. For the horizontal asymptote(s), we consider $\lim_{x\to\pm\infty} f(x)$. We have

$$\begin{aligned}\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{2x}{x^2-1}\\ &= \lim_{x\to\pm\infty} \frac{2x}{x^2-1}\left(\frac{1/x^2}{1/x^2}\right) \text{ dividing the numerator and}\\ &\quad \text{denominator by the highest power}\\ &\quad \text{of } x \text{ in the denominator}\end{aligned}$$

Exercise 2.6.70 (continued 1)

Solution (continued).

$$\begin{aligned}\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{(2x)/x^2}{(x^2-1)/x^2} = \lim_{x\to\pm\infty} \frac{2/x}{1-1/x^2} \text{ since } x\to\infty\\ &\quad \text{then we can assume that } x\neq 0\\ &= \frac{\lim_{x\to\pm\infty} 2/x}{\lim_{x\to\pm\infty}(1-1/x^2)} \text{ by the Quotient Rule}\\ &\quad \text{(Theorem 2.12(5)), assuming the denominator is not 0}\\ &= \frac{2\lim_{x\to\pm\infty} 1/x}{\lim_{x\to\pm\infty} 1 - (\lim_{x\to\pm\infty} 1/x)^2} \text{ by the Difference,}\\ &\quad \text{Constant Mult., and Power Rules, Theorem 2.12(2, 4, 6)}\\ &= \frac{2(0)}{1-(0)^2} = \frac{0}{1} = 0 \text{ by Example 2.6.1.}\end{aligned}$$

So $\boxed{y = 0 \text{ is a horizontal asymptote}}$ of $y = \dfrac{2x}{x^2-1}$.

Exercise 2.6.70 (continued 2)

Solution (continued). Now $f(x) = \dfrac{2x}{x^2-1} = \dfrac{2x}{(x+1)(x-1)}$ is a rational function with $\lim_{x\to -1} 2x = -2 \neq 0$, $\lim_{x\to -1} x^2 - 1 = 0$, and $\lim_{x\to 1} x^2 - 1 = 0$ (each by Theorem 2.2), so by Dr. Bob's Infinite Limits Theorem (applied to rational functions) $\boxed{f \text{ has vertical asymptotes at } x=-1 \text{ and } x=1}$. We explore the vertical asymptotes by taking one-sided limits to determine if the limit is $+\infty$ or $-\infty$. We analyze the sign of $\dfrac{2x}{x^2-1} = \dfrac{2x}{(x-1)(x+1)}$ for "appropriate" x in each case. For $x \to -1^+$, the appropriate x are close to -1 and slightly greater than -1. For such x, we have $2x$ is negative (in fact, $2x$ is "close to" -2), $x-1$ is negative (in fact, $x-1$ is "close to" -2), and $x+1$ is positive (since x is greater than -1; so $x+1$ is positive and "close to" 0). Combining the factors we get the sign diagram:
$\dfrac{2x}{x^2-1} = \dfrac{2x}{(x-1)(x+1)} \Rightarrow \dfrac{(-)}{(-)(+)} = +$. So $\boxed{\lim_{x\to -1^+} f(x) = \infty}$.

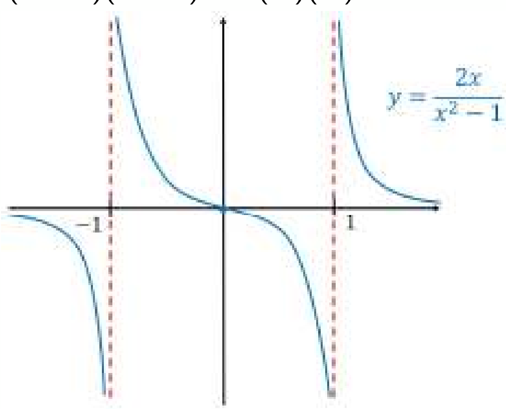
Exercise 2.6.70 (continued 3)

Solution (continued). For $x \to -1^-$, the appropriate x are close to -1 and slightly less than -1. For such x, we have $2x$ is negative ($2x$ is "close to" -2), $x-1$ is negative ($x-1$ is "close to" -2), and $x+1$ is negative (since x is less than -1; so $x+1$ is negative and "close to" 0). Combining the factors we get the sign diagram:
$\dfrac{2x}{x^2-1} = \dfrac{2x}{(x-1)(x+1)} \Rightarrow \dfrac{(-)}{(-)(-)} = -$. So $\boxed{\lim_{x\to -1^-} f(x) = -\infty}$.
For $x \to 1^+$, the appropriate x are close to 1 and slightly greater than 1. For such x, we have $2x$ is positive ($2x$ is "close to" 2), $x-1$ is positive (since x is greater than 1; so $x-1$ is positive and "close to" 0), and $x+1$ is positive ($x+1$ is "close to" 2). Combining the factors we get the sign diagram: $\dfrac{2x}{x^2-1} = \dfrac{2x}{(x-1)(x+1)} \Rightarrow \dfrac{(+)}{(+)(+)} = +$. So $\boxed{\lim_{x\to 1^+} f(x) = \infty}$.

Exercise 2.6.70 (continued 4)

Solution (continued). For $x \to 1^-$, the appropriate x are close to 1 and slightly less than 1. For such x, we have $2x$ is positive ($2x$ is "close to" 2), $x-1$ is negative (since x is less than 1; so $x-1$ is negative and "close to" 0), and $x+1$ is positive ($x+1$ is "close to" 2). Combining the factors we get the sign diagram: $\dfrac{2x}{x^2-1} = \dfrac{2x}{(x-1)(x+1)} \Rightarrow \dfrac{(+)}{(-)(+)} = -$. So $\boxed{\lim_{x\to 1^-} f(x) = -\infty}$.

We have the graph (notice the $\boxed{\text{range is all real numbers}}$):



$\square$

Example 2.6.20 (continued)

Solution (continued).

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\pm\infty} \left(1 - \frac{2}{3x} + \frac{3}{3x^2} - \frac{5}{3x^3} + \frac{6}{3x^4}\right) \\
&= \lim_{x\to\pm\infty} 1 - \frac{2}{3} \lim_{x\to\pm\infty} \frac{1}{x} + \left(\lim_{x\to\pm\infty} \frac{1}{x}\right)^2 - \frac{5}{3}\left(\lim_{x\to\pm\infty} \frac{1}{x}\right)^3 \\
&\quad +2\left(\lim_{x\to\pm\infty} \frac{1}{x}\right)^4 \text{ by the Sum, Difference,} \\
&\quad \text{Constant Multiple, and Power Rules,} \\
&\quad \text{Theorem 2.12(1, 2, 4, 6)} \\
&= 1 - \frac{2}{3}(0) + (0)^2 - \frac{5}{3}(0)^3 + 2(0)^4 = 1 \text{ by Example 2.6.1.}
\end{aligned}$$

Since the limit is 1, then g is a dominant term of f, as claimed. $\square$

Exercise 2.6.108 (again)

Exercise 2.6.108 (again). Consider the rational function $y = \dfrac{x^2-1}{2x+4}$. Find all asymptotes and graph in a way that reflects the asymptotic behavior.

Solution. We saw above that the graph of $y = \dfrac{x^2-1}{2x+4} = \dfrac{x}{2} - 1 + \dfrac{3}{2x+4}$ has $y = \dfrac{x}{2} - 1$ as an oblique asymptote as $x \to \pm\infty$. We now explore vertical asymptotes. By the Quotient Rule, Theorem 2.1(5), $\lim_{x\to c} f(x) = \lim_{x\to c} \dfrac{x^2-1}{2x+4} = \dfrac{c^2-1}{2c+4}$ for $c \neq -2$. So by definition, f is continuous on its domain $(-\infty,-2)\cup(-2,\infty)$. By Dr. Bob's Infinite Limits Theorem (applied to rational function f), since $\lim_{x\to-2} x^2-1 = (-2)^2-1 = 3 \neq 0$ and $\lim_{x\to-2} 2x+4 = x(-2)+4 = 0$ (by Theorem 2.2), we see that $\lim_{x\to-2^\pm} f(x) = \pm\infty$ and so the graph has a $\boxed{\text{vertical asymptote of } x = -2}$. We explore one-sided limits to see if the limits are ∞ or $-\infty$.

Exercise 2.6.108 (again, continued 1)

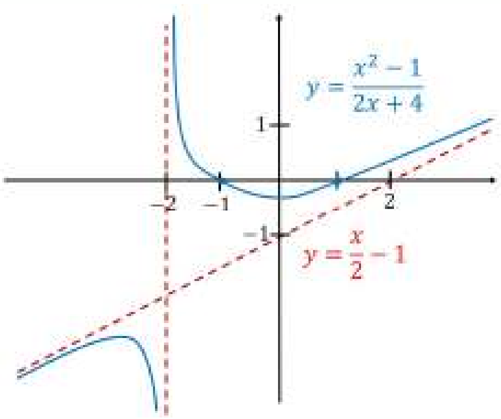
Solution (continued). For $\lim_{x\to-2^+} f(x)$, we analyze the sign of $\dfrac{x^2-1}{2x+4}$ for "appropriate" x (since $x \to -2^+$, then appropriate x are close to -2 and slightly greater than -2). For such x, we have x^2-1 is positive (in fact, x^2-1 is "close to" 3) and $2x+4$ is positive (since x is greater than -2; so $2x+4$ is positive and "close to" 0). Combining the factors we get the sign diagram: $\dfrac{x^2-1}{2x+4} \Rightarrow \dfrac{(+)}{(+)} = +$. So

$$\boxed{\lim_{x\to-2^+} f(x) = \lim_{x\to-2^+} \frac{x^2-1}{2x+4} = \infty}.$$

For $\lim_{x\to-2^-} f(x)$, we analyze the sign of $\dfrac{x^2-1}{2x+4}$ for "appropriate" x (since $x \to -2^-$, then appropriate x are close to -2 and slightly less than -2). For such x, we have x^2-1 is positive (in fact, x^2-1 is "close to" 3) and $2x+4$ is negative (since x is less than -2; so $2x+4$ is negative and "close to" 0).

Exercise 2.6.108 (again, continued 2)

Solution (continued). Combining the factors we get the sign diagram: $\dfrac{x^2-1}{2x+4} \Rightarrow \dfrac{(+)}{(-)} = -$. So $\boxed{\lim_{x\to-2^-} f(x) = \lim_{x\to-2^-} \dfrac{x^2-1}{2x+4} = -\infty}$. So the graph is:



□

Exercise 2.6.80

Exercise 2.6.80. Find a function g that satisfies the conditions $\lim_{x\to\pm\infty} g(x) = 0$, $\lim_{x\to3^-} g(x) = -\infty$, and $\lim_{x\to3^+} g(x) = \infty$. Graph $y = g(x)$ in a way that reflects the asymptotic behavior.

Solution. Since we want $\lim_{x\to\pm\infty} g(x) = 0$, then the graph of $y = g(x)$ will have $y = 0$ as a horizontal asymptote. Since we want $\lim_{x\to3^-} g(x) = -\infty$ and $\lim_{x\to3^+} g(x) = \infty$, then the graph of $y = g(x)$ has a vertical asymptote of $x = 3$. We try to find a rational function, $g(x) = p(x)/q(x)$, satisfying these conditions. If we make polynomial p of degree less than that of polynomial q, then this will give (as we will check) the horizontal asymptote $y = 0$. If we have $x - 3$ in the denominator then we should get a vertical asymptote of $x = 3$ (unless we also have a factor of $x - 3$ in the numerator, which we will avoid). So we try $p(x) = 1$ (a polynomial of degree 0), $q(x) = x - 3$ (a polynomial of degree 1), and $g(x) = 1/(x-3)$ (we may have to adjust the sign of g to get the proper one sided limits at 3).

Exercise 2.6.80 (continued 1)

Solution (continued). We have

$$\lim_{x\to\pm\infty} g(x) = \lim_{x\to\pm\infty}\frac{1}{x-3} = \lim_{x\to\pm\infty}\frac{1}{x-3}\left(\frac{1/x}{1/x}\right) \text{ dividing the numerator and denominator by the effective highest power of } x \text{ in the denominator}$$

$$= \lim_{x\to\pm\infty}\frac{1/x}{(x-3)/x} = \lim_{x\to\pm\infty}\frac{1/x}{1-3/x} \text{ since } x\to\pm\infty \text{ then we can assume that } x\neq 0$$

$$= \frac{\lim_{x\to\pm\infty} 1/x}{\lim_{x\to\pm\infty} 1 - 3\lim_{x\to\pm\infty} 1/x} \text{ by the Difference, Constant Mult., and Quotient Rules, Theorem 2.12(2,4,5)}$$

$$= \frac{(0)}{1-3(0)} = \frac{0}{1} = 0 \text{ by Example 2.6.1.}$$

So $y = 0$ and a horizontal asymptote of the graph of $y = g(x)$, as desired.

Exercise 2.6.80 (continued 2)

Solution (continued). Since $\lim_{x\to 3} 1 = 1 \neq 0$ and $\lim_{x\to 3} x - 3 = 0$ (both by Theorem 2.2, say), then by Dr. Bob's Infinite Limits Theorem (applied to rational functions) $\lim_{x\to 3^\pm} g(x) = \pm\infty$. We consider one-sided limits (as required by the question).

For $\lim_{x\to 3^+} g(x)$, we analyze the sign of $\dfrac{1}{x-3}$ for "appropriate" x (since $x\to 3^+$, then appropriate x are close to 3 and slightly greater than 3). For such x, we have 1 is positive and $x-3$ is positive (since x is greater than 3; so $x-3$ is positive and "close to" 0). Combining the factors we get the sign diagram: $\dfrac{1}{x-3} \Rightarrow \dfrac{(+)}{(+)} = +$. So $\boxed{\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+}\dfrac{1}{x-3} = \infty}$, as desired.

For $\lim_{x\to 3^-} g(x)$, we analyze the sign of $\dfrac{1}{x-3}$ for "appropriate" x (since $x\to 3^-$, then appropriate x are close to 3 and slightly less than 3). For such x, we have 1 is positive and $x-3$ is negative (since x is less than 3; so $x-3$ is negative and "close to" 0).

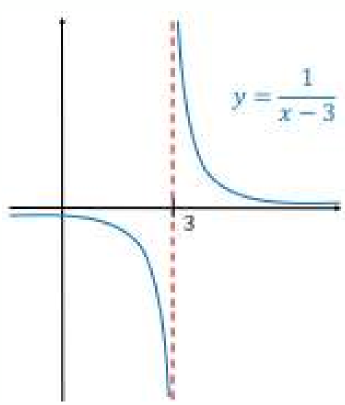
Exercise 2.6.80 (continued 3)

Exercise 2.6.80. Find a function g that satisfies the conditions $\lim_{x\to\pm\infty} g(x) = 0$, $\lim_{x\to 3^-} g(x) = -\infty$, and $\lim_{x\to 3^+} g(x) = \infty$. Graph $y = g(x)$ in a way that reflects the asymptotic behavior.

Solution (continued). Combining the factors we get the sign diagram: $\dfrac{1}{x-3} \Rightarrow \dfrac{(+)}{(-)} = -$. So

$\boxed{\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-}\dfrac{1}{x-3} = -\infty}$, as desired.

We then have the graph:



□

Exercise 2.6.102

Exercise 2.6.102. Use the formal definition of an infinite one-sided limit to prove that $\displaystyle\lim_{x\to 2^-}\frac{1}{x-2} = -\infty$.

Proof. Let f be a function defined on an interval (a, c), where $a < c$. We say that $f(x)$ *approaches negative infinity as x approaches c from the left*, and we write $\lim_{x\to c^-} f(x) = -\infty$, if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$c - \delta < x < c \text{ implies } f(x) < -B.$$

(This is the solution to Exercise 2.6.99(c).) Notice that $f(x) = 1/(x-2)$ is defined on the interval $(-\infty, 2)$ (here, $c = 2$). Let $-B$ be any negative real number. Choose $\delta = 1/B > 0$. If $2-\delta < x < 2$ then $-\delta < x - 2 < 0$ and $-1/\delta > 1/(x-2)$, since $1/x$ is a decreasing function for negative input values. Now $\delta = 1/B$ so $1/\delta = B$ and $-1/\delta = -B$. So $2-\delta < x < 2$ implies $f(x) = 1/(x-2) < -B$. Therefore, by the definition above, $\displaystyle\lim_{x\to 2^-} 1/(x-2) = -\infty$. □